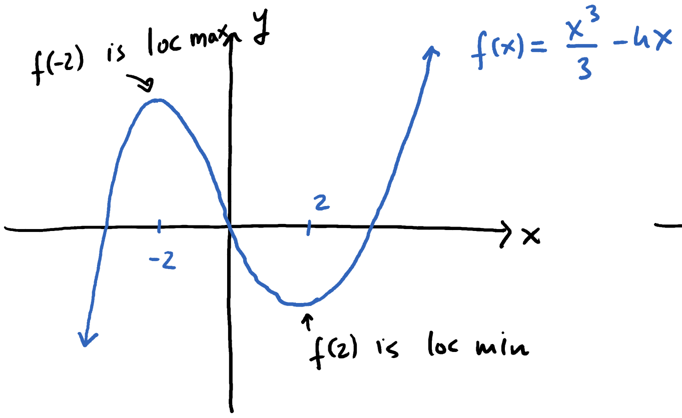
Absolute Maxima and Minima

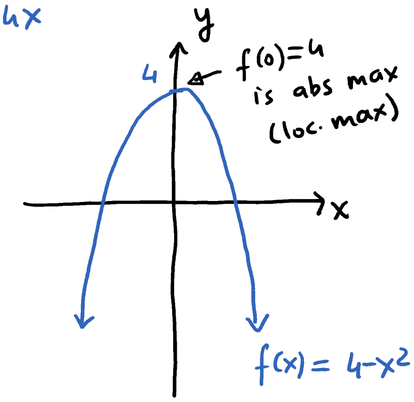
Recall. For x near to c; if $f(x) \le f(c)$ we say that $f(c)$ is local max and if $f(x) \geq f(c)$ we say that $f(c)$ is a loc min.

Defn. If $f(c) \geq f(x)$ for all x in the domain of f, then $f(c)$ is called absolute maximum of f. If $f(c) \le f(x)$ for all x in the domain of f, then $f(c)$ is called absolute minimum of f.

* Abs max or min is called an abs. extremum.

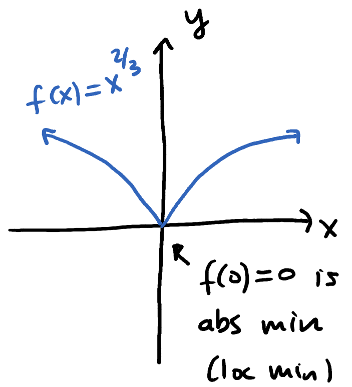


No absolute extrema



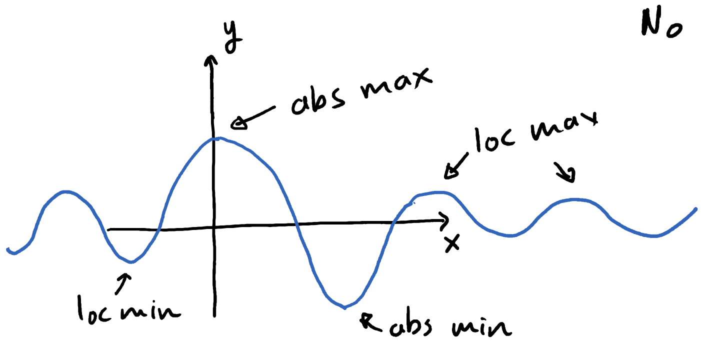
$f(0)=4$ is abs max.

No absolute minimum



$f(0)=0$ is abs min.

No abs. max

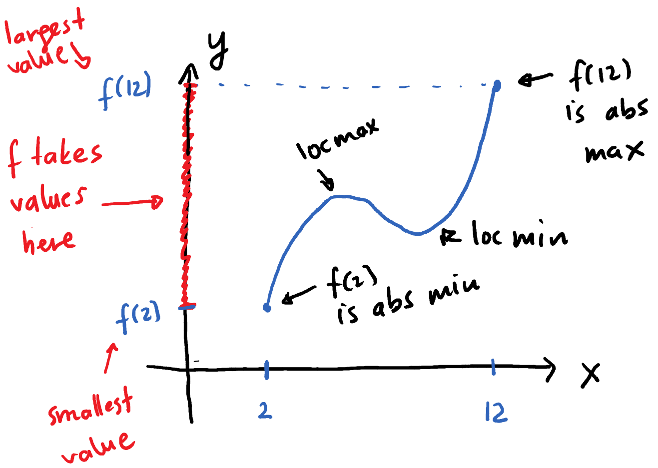


Question. Under which conditions a func. must have abs. extremas?

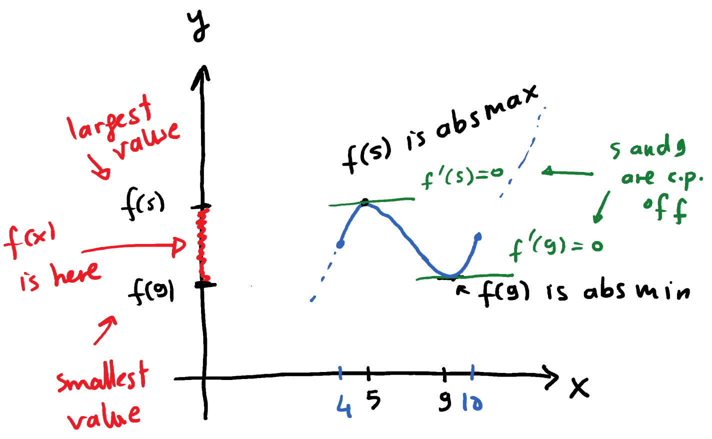
Teorem. A func. f that is continuous on a closed interval $[a,b]$ has both an absolute maximum an absolute minimum on $[a,b]$

i) Interval is closed and bdd : $[a,b]$ ii) f is cont on $[a,b]$

Note. Absolute extremas depend on both the func f and the interval $[a,b]$. See the following graphs:



$[a,b]=[2,12]$



$[a,b]=[4,10]$

Note. Both absolute maximum and the absolute minimum are unique, but each can occur at more than one point in the interval.

Question. If they exist, where absolute extrema occur?

Thm. Absolute extrema (if they exist) must occur at critical numbers or at endpoints.

Finding Absolute Extrema on a Closed Interval:

1) Identify endpoints and critical numbers in the interval.

2) Evaluate the func. at each.

3) Choose the largest and smallest values.

Ex. Find the absolute extrema of

$$f(x) = x^3+3x^2-9x-7$$

on each of the following intervals:

a) $[-6,4]$ b) $[-4,2]$ c) $[-2,2]$

Soln. Since f is polynomial, it is cont for all values of x. This means that f takes abs extremas on each closed intervals.

Find c.p. of f: $f'(x) = 3x^2+6x-9 = 3(x^2+2x-3) = 3(x+3)(x-1)$

So, $x=-3$ and $x=1$ are critical points of f:

a) $[-6,4]$; c.p are -3 and 1

end points
$f(-6) = -61$ ← abs min
$f(4) = 69$ ← abs max

c.p. of f
$f(-3) = 20$
$f(1) = -12$

f takes abs. extrema at endpoints.

b) $[-4,2]$; c.p : -3 and 1

end points
$f(-4) = 13$
$f(2) = -5$

c.p. of f
$f(-3) = 20$ ← abs max
$f(1) = -12$ ← abs min

f takes abs. extrema at critical points

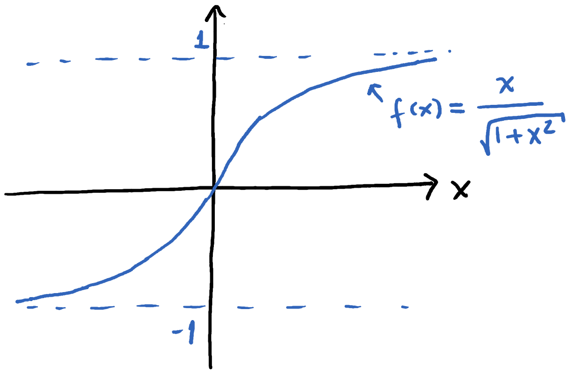
c) $[-2,2]$; c.p. 1 ; $-3 \notin [-2,2]$

end points
$f(-2) = 15$ ← abs max
$f(2) = -5$

c.p. of f
$f(1) = -12$ ← abs min

f takes abs max at end point and abs min at c.p.

Remark. If f is not cont. or the interval is not closed or bounded, then f may fail to has absolute extrema.



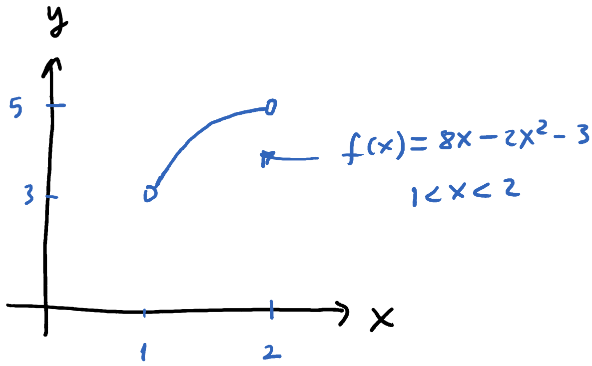
Unbounded interval $(-\infty, \infty)$.

No abs. extrema on $(-\infty, \infty)$

$-1 < f(x) < 1$ for all x

For any x $f(x) \neq 1$ or -1.

$\lim_{x\to\infty} f(x) = 1$ & $\lim_{x\to-\infty} f(x) = -1$.

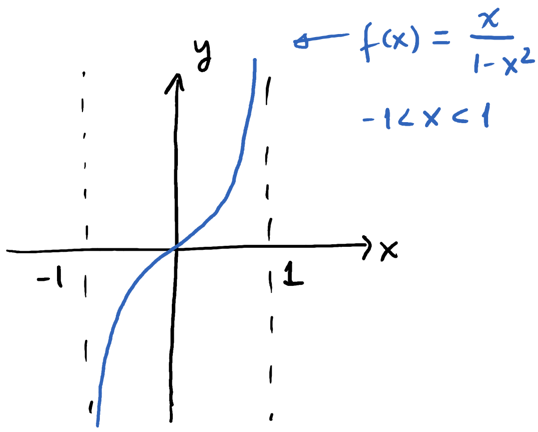


Open interval $(1,2)$.

No abs. extrema on $(1,2)$.

$3 < f(x) < 5$ for all $x \in (1,2)$.

For any $x \in (1,2)$; $f(x) \neq 3$ or 5



Open interval $(-1,1)$

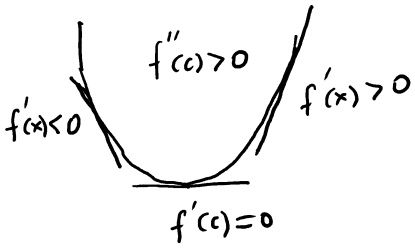
No absolute extrema on $(-1,1)$.

Graph has vertical asy at $x=-1$ and $x=1$.

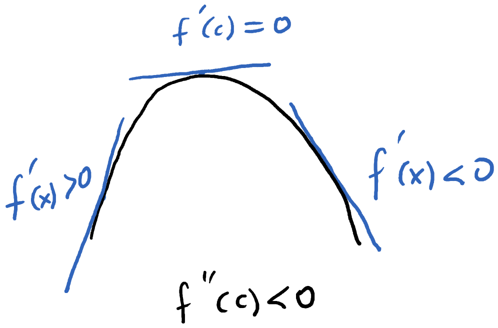
$$\lim_{x\to 1^-} f(x) = \infty$$

$$\lim_{x\to -1^+} f(x) = -\infty$$

Second Derivative and Extrema



$f'(c)=0$ and $f''(c)>0$ implies $f(c)$ is a loc min.



$f'(c)=0$ and $f''(c)<0$ implies $f(c)$ is a loc max.

Result (Second Derivative Test)

Let c be a critical number of $f(x)$ such that $f'(c)=0$. If the second derivative $f''(c)>0$, then $f(c)$ is a local minimum. If $f''(c)<0$, then $f(c)$ is a local maximum.

Summary

$f'(c)$	$f''(c)$	Graph of f	$f(c)$	Example
0	+	Concave Upward	Loc. Min	
0	−	Concave downward	Loc Max	
0	0	?	No info	

When, $f'(c)=0$ and $f''(c)=0$ the test does not work. Hence we should apply First Derivative Test.

Recall (First Derivative Test)

x		c	
f'	−	0	+
f	↘		↗

$f(c)$ loc min.

x		c	
f'	+	0	−
f	↗		↘

$f(c)$ is a loc max

x		c	
f'	±	0	±
f	↗ →		↗ →

No loc. extrema

EXAMPLE 2 Testing Local Extrema Find the local maxima and minima for each function. Use the second-derivative test for local extrema when it applies.

(A) $f(x) = x^3 - 6x^2 + 9x + 1$

(B) $f(x) = xe^{-0.2x}$

(C) $f(x) = \frac{1}{6}x^6 - 4x^5 + 25x^4$

Soln. a) $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-3)(x-1)$

$f'(x) = 0 \Rightarrow x = 3$ and $x = 1$ are c.p. of f

$f''(x) = 6x - 12$; $f''(3) = 18 - 12 = 6 > 0 \Rightarrow f(3)$ is a loc min

$f''(1) = 6 - 12 = -6 < 0 \Rightarrow f(1)$ is a loc max

b) $f(x) = xe^{-0.2x}$

$f'(x) = 1e^{-0.2x} + (e^{-0.2x} \cdot (-0.2)\, x) = e^{-0.2x}(1 - 0.2x)$

$f'(x) = 0 \Rightarrow 1 - 0.2x = 0 \Rightarrow x = 5$ is the only c.p. of f

$$f''(x) = -0.2\, e^{-0.2x} - 0.2\, e^{-0.2x}(-0.2x) + (-0.2)\, e^{-0.2x}$$
$$= -0.2\, e^{-0.2x} + 0.04x\, e^{-0.2x} - 0.2\, e^{-0.2x}$$
$$= e^{-0.2x}(0.04x - 0.4)$$

$f''(5) = e^{-0.2(5)}(0.2 - 0.4) = \frac{-0.2}{e^1} < 0$

So $f(5)$ is a loc max.

c) $f(x) = \frac{1}{6}x^6 - 4x^5 + 25x^4$

$f'(x) = x^5 - 20x^4 + 100x^3 = x^3(x^2 - 20x + 100) = x^3(x-10)^2$

$f'(x) = 0 \Rightarrow x = 0$ and $x = 10$ are c.p. of f.

$f''(x) = 5x^4 - 80x^3 + 300x^2$

$f''(0) = 0$ and $f''(10) = 0 \Rightarrow$ Second Derivative Test does not applicable.

x	$-\infty$		0		10		∞
f'		$-$	0	$+$	0	$+$	
f		↘		↗		↗	

$f(0)$ is loc min but at $x = 10$ f does not have any loc. extrema.

Note that if the func. in queostion has only one critical number then second derivative test for local extrema not only classifies the local extremum but also guarantees that the local extremum is, in fact, the absolute extremum.

Thm. Let f be cont. on an open interval I with only one critical number c in I. Then

i) If $f'(c)=0$ and $f''(c)>0$ then $f(c)$ is the absolute minimum of f on I.

ii) If $f'(c)=0$ and $f''(c)<0$ then $f(c)$ is the absolute maximum of f on I.

iii) When $f'(c)=0$ and $f''(c)=0$; the test gives no information.

Ex. Find the absolute extrema of each func. on $(0,\infty)$.

a) $f(x) = x + \frac{4}{x}$ b) $f(x) = (\ln x)^2 - 3\ln x$

Soln a) $f'(x) = 1 - \frac{4}{x^2} = 0 \Rightarrow 1 = \frac{4}{x^2} \Rightarrow x^2 = 4 \Rightarrow x = 2$

$-2 \notin (0,\infty)$

f has only one critical point: $x=2$.

$f''(x) = \frac{8}{x^3}$; $f''(2) = \frac{8}{8} = 1 > 0$

Since $f''(2) > 0$ and $x=2$ is the only c.p. of f;

$f(2)=4$ is the absolute minimum of f on $(0,\infty)$.

b) $f(x) = (\ln x)^2 - 3\ln x$

$$f'(x) = 2\ln x\left(\frac{1}{x}\right) - 3\frac{1}{x} = \frac{1}{x}(2\ln x - 3)$$

$$f'(x) = 0 \Rightarrow \frac{1}{x}(2\ln x - 3) = 0 \Rightarrow 2\ln x - 3 = 0$$

$$\Rightarrow \ln x = \frac{3}{2}$$

$$\Rightarrow e^{\ln x} = e^{3/2}$$

$$\Rightarrow x = e^{3/2}$$ is the only c.p. of f.

$$f''(x) = -\frac{1}{x^2}(2\ln x - 3) + \frac{2}{x}\cdot\frac{1}{x}$$

$$= \frac{1}{x^2}(3 - 2\ln x + 2) = \frac{1}{x^2}(5 - 2\ln x)$$

$$f''(e^{3/2}) = \frac{1}{e^3}(5 - 2\ln e^{3/2}) = \frac{1}{e^3}\left(5 - \cancel{2}\,\frac{3}{\cancel{2}}\right) = \frac{2}{e^3} > 0$$

Since $x = e^{3/2}$ is the only c.p. of f and $f''(e^{3/2}) = \frac{2}{e^3} > 0$, $f(e^{3/2})$ is the absolute minimum of f on $(0,\infty)$.

Matched Problem. Find the absolute extrema of each function on $(0,\infty)$.

a) $f(x) = 12 - x - \frac{5}{3}$ b) $f(x) = 5\ln x - x$.

Exercise.