Elasticity of Demand

Question. When will a price lead to an increase in revenue?

Economists use the notion of elasticity of demand to answer this question and study relationships among price, demand and revenue.

Relative Rate of Change

Defn. The relative rate of change of a func. $f(x)$ is defined by $f'(x)/f(x)$, or equivalently, $\frac{d}{dx}\ln f(x)$.

The percentage rate of change is $100 * \frac{f'(x)}{f(x)}$, or equivalently $100 * \frac{d}{dx}\ln f(x)$.

Ex. Find the relative and percentage rate of change of $f(x) = 3000 - 8x^2$ at $x = 18$.

Soln. $f'(x) = -16x \Rightarrow f'(x)/f(x) = \dfrac{-16x}{3000 - 8x^2}$

$$\Rightarrow \frac{f'(18)}{f(18)} = \frac{-16(18)}{3000 - 8(18)^2} = \frac{-288}{408}$$

relative rate of change of f at $x=18$ $= -0.705$

So, percentage rate of change is equal to $-0.705(100) = -7.05\%$

$$\text{Elasticity of demand} = -\frac{\text{relative rate of change of demand}}{\text{relative rate of change of price}}$$

$x = f(p)$, p - price, x - demand

demand is function of price

Elasticity of demand at price p is denoted by $E(p)$.

$$E(p) = -\frac{\frac{d}{dp}\ln f(p)}{\frac{d}{dp}\ln p}$$

$$= -\frac{f'(p)/f(p)}{1/p} = \frac{-p\,f'(p)}{f(p)}$$

Teorem. If price and demand are related by $x = f(p)$, then the elasticity of demand is given by

$$E(p) = -\frac{p\,f'(p)}{f(p)}.$$

Why the formula has a minus sign:

$$\underbrace{E(p)}_{+} = \overset{\uparrow}{\underset{\text{put minus}}{-}}\ \underbrace{\frac{\overset{>0}{p}\ \overset{<0}{f'(p)}}{\underset{>0}{f(p)}}}_{-}$$

$E(p)$	Demand	Interpretation
$0 < E < 1$	inelastic	Demand is not sensitive to changes in price, i.e. percentage change in Price produces a smaller percentage change in demand.
$E > 1$	elastic	Demand is ~~not~~ sensitive to changes in price, i.e. percentage change in Price produces a larger percentage change in demand.
$E = 1$	unit	A percentage change in price produces the same percentage change in demand.

EXAMPLE 2 Elasticity of Demand The price p and the demand x for a product are related by the price–demand equation

$$x + 500p = 10{,}000 \tag{1}$$

Find the elasticity of demand, $E(p)$, and interpret each of the following:

(A) $E(4)$ (B) $E(16)$ (C) $E(10)$

Soln. $x + 500p = 10\,000 \Rightarrow x = 10\,000 - 500p := f(p)$

$$E(p) = -\frac{p\,f'(p)}{f(p)} \quad ; \quad f'(p) = -500$$

$$= -\frac{p(-500)}{10\,000 - 500p} = \frac{\cancel{500}\,p}{\cancel{500}(20-p)} = \frac{p}{20-p}$$

For any p ; $E(p) = \dfrac{p}{20-p}$.

a) $E(4) = \dfrac{4}{20-4} = \dfrac{4}{16} = 0.25 < 1 \Rightarrow$ demand is inelastic

$p \uparrow \downarrow \%10 \Rightarrow x \downarrow \uparrow (0.25)\,10 = 2.5\,\%$

b) $E(16) = \dfrac{16}{20-16} = \dfrac{16}{4} = 4 > 1 \Rightarrow$ demand is elastic

$p \uparrow \downarrow \%10 \Rightarrow x \downarrow \uparrow 4(10) = 40\%$

c) $E(10) = \dfrac{10}{20-10} = 1 = 1 \Rightarrow$ demand is unit

$p \uparrow \downarrow \%10 \Rightarrow x \downarrow \uparrow \%10$

Matched Problem 2 Find $E(p)$ for the price–demand equation

$$x = f(p) = 1{,}000(40 - p)$$

Find and interpret each of the following:

(A) $E(8)$ (B) $E(30)$ (C) $E(20)$

Soln. $f(p) = 1000(40-p) \Rightarrow f'(p) = -1000$

Hence

$$E(p) = -\frac{p f'(p)}{f(p)} = -\frac{p(-1000)}{1000(40-p)} = \frac{p}{40-p}$$

a) $E(8) = \dfrac{8}{40-8} = \dfrac{8}{32} = 0.25 < 1 \Rightarrow$ demand is inelastic

This yields that

$$p \uparrow / \downarrow \ \% 10 \Longrightarrow x \downarrow / \uparrow \ \% (0.25)(10) = 2.5$$

b) $E(30) = \dfrac{30}{40-30} = 3 > 1 \Rightarrow$ emand is elastic.

It follows that

$$p \uparrow / \downarrow \ \% 10 \Longrightarrow x \downarrow / \uparrow \ \% (3)(10) = 30$$

c) $E(20) = \dfrac{20}{40-20} = 1 = 1 \Longrightarrow$ demand is unit.

Hence,

$$p \uparrow / \downarrow \ \% 10 \Longrightarrow x \downarrow \uparrow \ \% 10$$

Revenue and Elasticity: $x = f(p)$ price-demand eqn

$$R(p) = xp = p\,f(p)$$

$$R'(p) = 1\,f(p) + p\,f'(p) \qquad \text{“We take derivative wrt } p\text{”}$$

$$R'(p) = f(p) + \frac{p\,f'(p)}{f(p)}\cdot f(p)$$

$$R'(p) = f(p)\left[1 + \underbrace{\frac{p\,f'(p)}{f(p)}}_{-E(p)}\right] = \overset{>0}{f(p)}\left[1 - E(p)\right]$$

So, it follows that $R'(p)$ and $1 - E(p)$ have the same sign.

- $E(p) = 1 \implies R'(p) = 0 \implies R(p)$ is constant.
- $E(p) < 1 \implies R'(p) > 0 \implies R$ is increasing func. of p.
- $E(p) > 1 \implies R'(p) < 0 \implies R$ is decreasing func. of p.

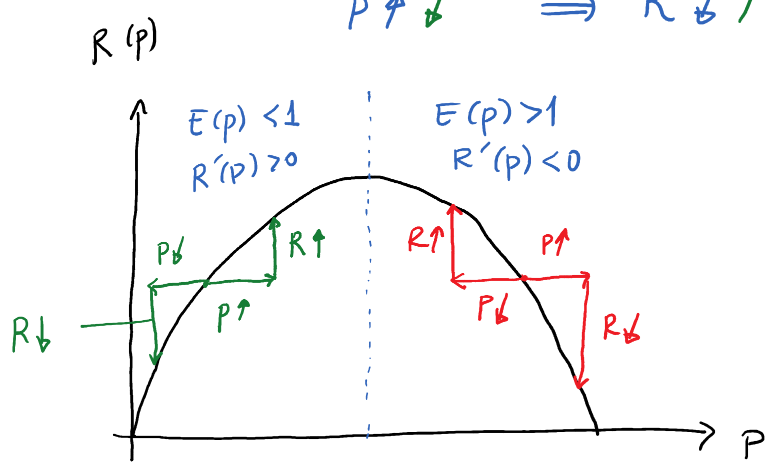
To summarize;

- If $E(p) < 1$, namely demand is inelastic, then

$$p \uparrow \downarrow \implies R \uparrow \downarrow$$

- If $E(p) > 1$, namely demand elastic, then

$$p \uparrow \downarrow \implies R \downarrow \uparrow$$



Relation between revenue and elasticity

EXAMPLE 3 Elasticity and Revenue A manufacturer of sunglasses currently sells one type for \$15 a pair. The price p and the demand x for these glasses are related by

$$\boxed{x = f(p) = 9{,}500 - 250p}$$

If the current price is increased, will revenue increase or decrease?

Matched Problem 3 Repeat Example 3 if the current price for sunglasses is \$21 a pair.

Soln. $f'(p) = -250$ so $E(p) = -\dfrac{p(-250)}{9500-250p}$

$$= \frac{\cancel{250}\,p}{\cancel{250}(38-p)} = \frac{p}{38-p}$$

a) $p = \$15$; $E(15) = \dfrac{15}{38-15} = \dfrac{15}{23} < 1$

So, demand is inelastic.

$$R = x \cdot p \quad \uparrow$$

$\%\left(\frac{15}{23}10\right) \downarrow \quad \uparrow \%10$

Revenue will increase when price increase.

b) $p = \$21$; $E(21) = \dfrac{21}{38-21} = \dfrac{21}{17} > 1$

So, demand is elastic.

$$R = x \quad p \quad \downarrow$$

$\%\left(\frac{21}{17}10\right) \downarrow \quad \uparrow \%10$

Revenue will decrease when price increase.

Question. When we maximize our Revenue?

At unit elasticity.

Ex. **Revenue and elasticity.** The price–demand equation for hamburgers at a fast-food restaurant is

$$x + 400p = 3{,}000$$

Currently, the price of a hamburger is \$3.00. If the price is increased by 10%, will revenue increase or decrease?

b) If the current price of a hamburger is \$4.00, will a 10 % price increase cause revenue to increase or decrease?

c) What price will maximize the revenue from selling hamburger?

Soln a) $x + 400p = 3000 \Rightarrow x = f(p) = 3000 - 400p$

$f'(p) = -400$; $E(p) = -\dfrac{p(-400)}{3000-400p} = \dfrac{\cancel{400}\,p}{\cancel{400}(7.5-p)}$

$= \dfrac{p}{7.5-p}$

At $p=3$, $E(3) = \dfrac{3}{7.5-3} = \dfrac{3}{4.5} < 1 \Rightarrow$ demand is inelastic

$\Rightarrow$ when price increase revenue will increase

b) At $p = 4$, $E(4) = \dfrac{4}{7.5-4} = \dfrac{4}{3.5} > 1 \Rightarrow$ demand is elastic

$\Rightarrow$ when price increase revenue will increase

c) For which value of p $E = 1$?

$E(p) = \dfrac{p}{7.5-p} = 1 \Rightarrow p = 7.5 - p$

$\Rightarrow 2p = 7.5$

$\Rightarrow p = 3.75\ \$$.

Hence, at $p = 3.75\ \$$, revenue is maximum.

Ex. Given the price–demand equation

$$p + 0.005x = 30$$

(A) Express the demand x as a function of the price p.

(B) Find the elasticity of demand, $E(p)$.

(C) What is the elasticity of demand when $p = \$10$? If this price is increased by 10%, what is the approximate percentage change in demand?

(D) What is the elasticity of demand when $p = \$25$? If this price is increased by 10%, what is the approximate percentage change in demand?

(E) What is the elasticity of demand when $p = \$15$? If this price is increased by 10%, what is the approximate percentage change in demand?

Ex. Use the price-demand equation

$$x = f(p) = \sqrt{2500 - 2p^2}$$

to find the values of p for which demand is elastic and the values for which demand is inelastic.

Ex. Use the demand eqn $x = f(p) = 20(10-p)$ to find the revenue function. Sketch the graph of the revenue func. and indicate the regions of inelastic and elastic demand on graph.