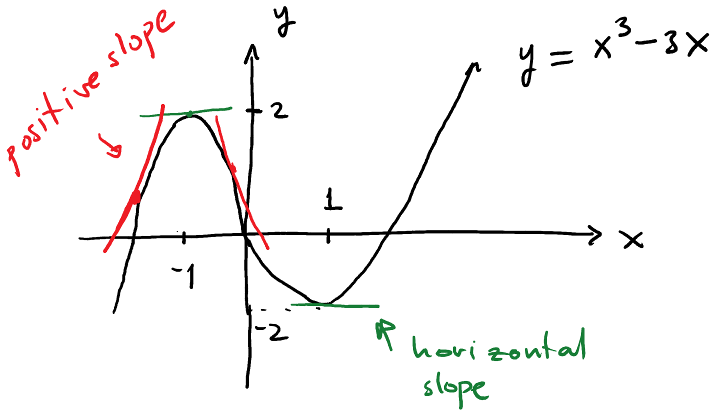
5. GRAPHING and OPTIMIZATION

5.1. First Derivative and Graphs:



- On the interval $(-\infty, -1)$ $(1, \infty)$ the graph of f is rising, f is increasing, tangent lines have positive slope ($f'(x) > 0$)
- On the interval $(-1, 1)$ the graph of f is falling, f is decreasing, tangent lines have negative slope ($f'(x) < 0$)
- At $x = -1$ and $x = 1$, the slope of the graph is zero ($f'(x) = 0$).

Defn. We say that the func. f is increasing on an interval I if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$, and f is decreasing on I if $f(x_2) < f(x_1)$ whenever $x_2 > x_1$.

Thm. For the interval (a, b) if $f'(x) > 0$ then f is increasing and $f'(x) < 0$, then f is decreasing.

$f'(x)$	$f(x)$	Graph of f	Examples
$+$	Increases ↗	Rises ↗	
$-$	decreases ↘	Falls ↘	

Ex. Let $f(x) = 8x - x^2$.

a) Which values of x correspond to horizontal tangent line.

b) For which values of x is $f(x)$ decreasing? Increasing?

c) Sketch a graph of f. Add any horizontal tangent line.

Soln.

a) Horizontal tangent line   Slope (m) = 0 ; $f'(x) = 0$?

$$f(x) = 8x - x^2 \implies f'(x) = 8 - 2x = 0 \implies x = 4$$

So, at $x=4$ the graph has horizontal tangent line.

b)

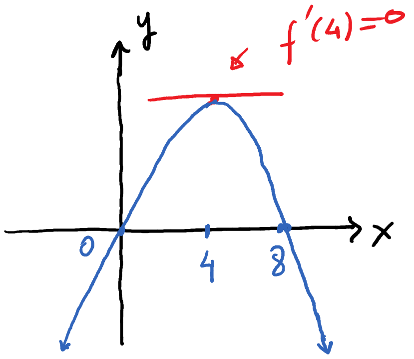
x	$-\infty$	4	∞
f'	$+$	0	$-$
f	$\nearrow$		$\searrow$

$f'(0) = 8 > 0$

$f'(5) = -2 < 0$

f is increasing on $(-\infty, 4)$ and decreasing on $(4, \infty)$.

c) $f(x) = 8x - x^2$



← f is increasing →← f is decreasing →

x-int. ; $8x - x^2 = 0 \implies x(8-x) = 0$
$\implies x = 0, \; x = 8$

y-int ; $x = 0 \implies y = 0$

Defn (Critical Number) A real number x in the domain of f such that $f'(x) = 0$ or $f'(x)$ d.n.e. is called a critical number of f.

Ex. For the following funcs, find the critical points, the interval on which f is increasing, and those on which f is decreasing?

a) $f(x) = 1+x^3$ b) $f(x) = (1-x)^{1/3}$ c) $f(x) = \frac{1}{x-2}$ d) $f(x) = 8\ln x - x^2$

Soln

a) $f(x) = 1+x^3$; Since f is polynomial, $\text{Dom} f = \mathbb{R}$.

i) $f'(x) = 3x^2$; $f'(x) = 0 \Rightarrow 3x^2 = 0 \Rightarrow x = 0$ c.n. of f

ii) $f'(x) = 3x^2$ is defined everywhere.

$\therefore$ $x=0$ is the only c.n. of f.

x	$-\infty$	0 (c.n.)	∞
f'	$+$	0	$+$
f	$\nearrow$		$\nearrow$

f is increasing everywhere and decreasing nowhere.

b) $f(x) = (1-x)^{1/3} = \sqrt[3]{1-x}$; $\text{Dom} f = \mathbb{R}$

i) $f'(x) = \frac{1}{3}(1-x)^{-2/3} \cdot (-1) = \dfrac{-1}{3(1-x)^{2/3}}$

$f'(x) = 0 \Rightarrow \dfrac{-1}{3(1-x)^{2/3}} = 0 \Rightarrow$ No soln

ii) $f'(1)$ is undefined and $f(1)$ is defined. So $x=1$ is a c.n. of f.

$\therefore$ $x=1$ is the only c.n. of f.

x	$-\infty$	1	∞
f'	$-$	N/D	$-$
f	$\searrow$		$\searrow$

$f'(0) = \dfrac{-1}{3(1-0)^{2/3}} = \dfrac{-1}{3} < 0$

$f'(2) = \dfrac{-1}{3(1-2)^{2/3}} = \dfrac{-1}{3} < 0$

f is decreasing on $\mathbb{R}$.

$(-1)^{2/3} = \sqrt[3]{(-1)^2} = \sqrt[3]{1} = 1$

c) $f(x) = \dfrac{1}{x-2}$; $\text{Dom} f = \mathbb{R} \setminus \{2\}$

i) $f(x) = (x-2)^{-1} \Rightarrow f'(x) = -(x-2)^{-2} = \dfrac{-1}{(x-2)^2}$

$f'(x) = 0 \Rightarrow \dfrac{-1}{(x-2)^2} = 0 \Rightarrow$ No soln

ii) $f'(2)$ is undefined but $f(2)$ is not defined also
So, $x=2$ is not a c.n. of f.

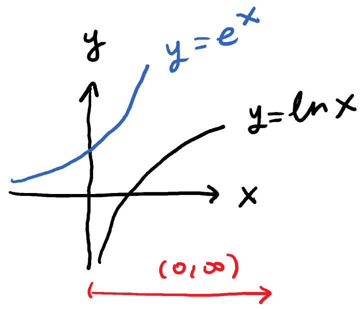
$\therefore$ There is no c.n. for f.

← f is not defined

x	$-\infty$	2	∞
f'	$-$	N/D	$-$
f	↘	N/D	↘

f is decreasing on $(-\infty, 2)$ and $(2, \infty)$. Not write $(-\infty, \infty)$.

d) $f(x) = 8\ln x - x^2$; $\text{Dom} f = (0, \infty)$



i) $f'(x) = 8\dfrac{1}{x} - 2x = 0 \Rightarrow \dfrac{8}{x} = 2x$

$\Rightarrow x^2 = 4 \Rightarrow x = 2$ or $\cancel{x=-2}$

not in the domain

$x=2$ is a c.n. of f.

ii) $f'(0)$ not defined but $f(0)$ is also not defined. Hence
$x=0$ is not c.n. of f.

← c.n.

x	0	2	∞
f'	$+$	0	$-$
f	↗		↘

$f'(x) = \dfrac{8}{x} - 2x$

$f'(1) = 6 > 0$

$f'(3) = \dfrac{8}{3} - 6 < 0$

f is ↗ on $(0,2)$ and ↘ on $(2, \infty)$.

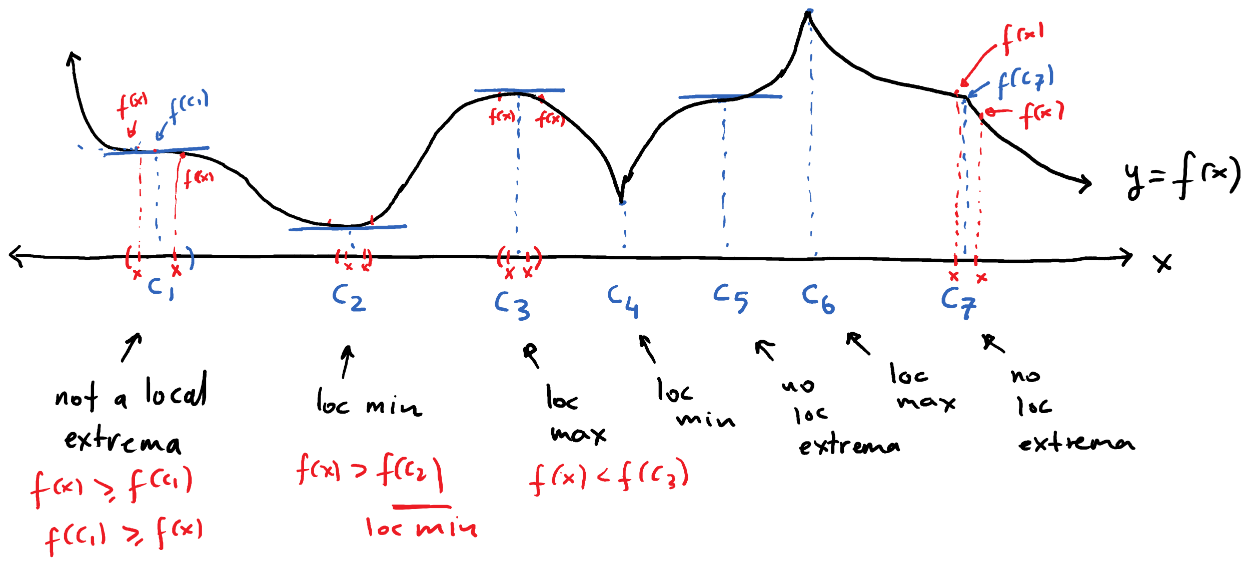
Local Extrema :

Defn. $f(c)$ is called local maximum (local minimum) if there exists an interval (m,n) containing c such that

$$f(x) \leq f(c) \quad \text{for all } x \text{ in } (m,n).$$

$$\left(f(x) \geq f(c) \quad \text{for all } x \text{ in } (m,n)\right)$$

* Note that this inequality need hold only for numbers x near c, which is why we use the term "local".



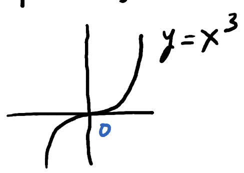
Question. How can we locate local maxima and minima if we are given the eqn of func. not its graph?

Thm. If $f(c)$ is a local extremum of the func. f, then c is a critical number of f.

Remark. Thm states that a local extremum can occur only at critical point ($f'(c)=0$ or $f'(c)$ undefined), but it does not imply that every critical point produces a loc. extr.

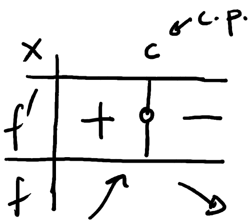
e.g. $f(x) = x^3$; $f'(x) = 3x^2 = 0 \Rightarrow x=0$ c.p. $f'(0)=0$.

but at $x=0$ there is no loc. extrema

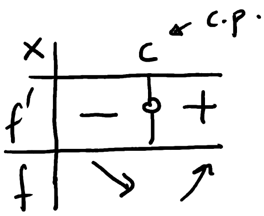


First Derivative Test: Let c be a critical point of f.

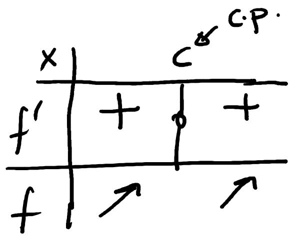
If $f'(c)$ changes in sign at c, then $f(c)$ is a local extremum.



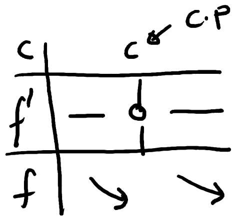
$(c, f(c))$ loc max



$(c, f(c))$ loc min

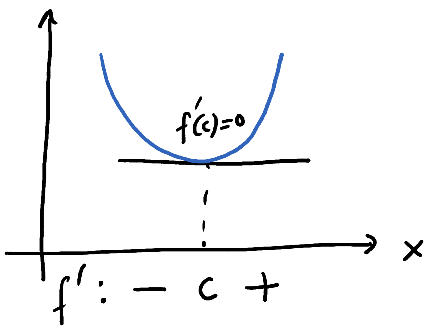


no loc extrema

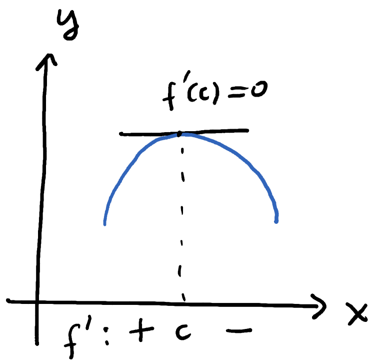


no loc. extrema

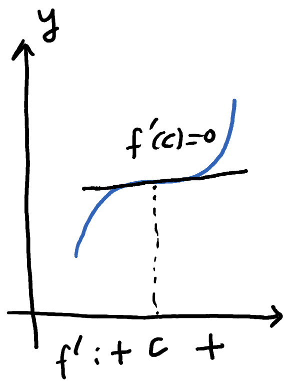
Some Examples.



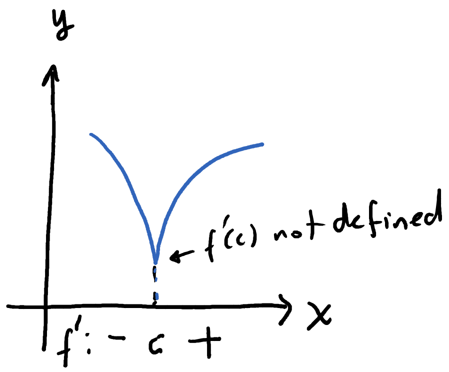
$f(c)$ loc min



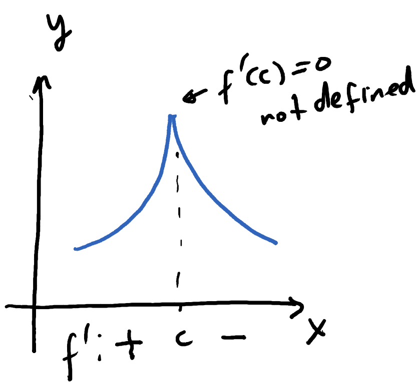
$f(c)$ loc max



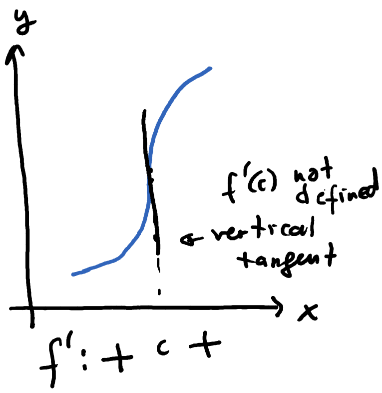
no loc ext.



$f(c)$ is a loc min



$f(c)$ is a loc max



no loc ext.

Ex. Let $f(x) = x^3 - 6x^2 + 9x + 1$.

a) Find the critical numbers of f.

b) Find the local maxima and minima of f.

c) Sketch a graph of f.

Soln. a) $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-3)(x-1)$

i) $f'(x) = 0 \Rightarrow x = 3$ or $x = 1$ c.p. of f.

ii) $f'(x)$ is defined everywhere $\Rightarrow$ no c.p. from here

b)

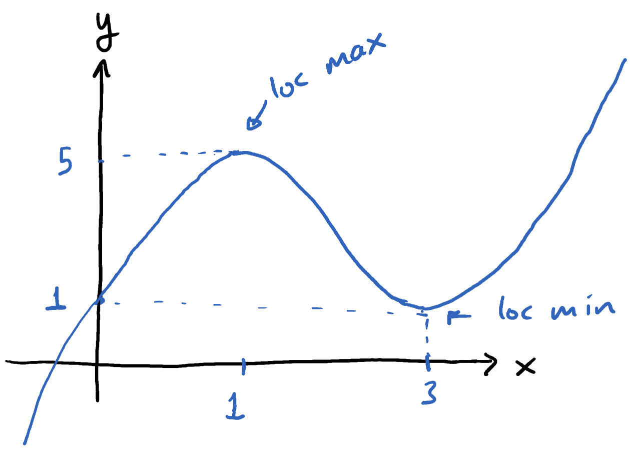
c.p.

x	$-\infty$		1		3		∞
f'		$+$	0	$-$	0	$+$	
f		$\nearrow$	$f(1)=5$ loc max	$\searrow$	$f(3)=1$ loc min	$\nearrow$	

f is increasing on $(-\infty, 1)$ and $(3, \infty)$ and decreasing $(1, 3)$.

$f(1)$ is a loc max and $f(3)$ is loc min.

c)



$f(x) = x^3 - 6x^2 + 9x + 1$

$f(0) = 1$

$\longleftarrow$ f is increasing — f is decreasing — f is increasing $\longrightarrow$

Ex. Let $f(x) = x\ln x - x$.

a) Find the critical numbers of f.

b) Find the local maxima and minima of f.

Soln.

a) Note that, since $\ln x$ is defined for $x>0$ domain of $f(x) = x\ln x - x$ is $(0,\infty)$. Now find c.p. if there is any

$$f'(x) = 1\ln x + x\cdot\frac{1}{x} - 1 = \ln x$$

i) $f'(x) = 0 \Rightarrow \ln x = 0 \Rightarrow x = e^0 = 1$ c.p.

ii) $f'(x)$ defined on $(0,\infty)$. Hence no c.p. from this case.

$\therefore$ There is only one c.p. which is $x=1$.

b) Make a sign chart for f':

c.p.

x	0	1	∞
f'	$-$	0	$+$
f	$\searrow$		$\nearrow$

$(1, f(1))$ loc min.

$f'(\tfrac{1}{2}) = \ln(\tfrac{1}{2}) < 0$

$f'(e) = \ln e = 1 > 0$

f is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

$(1, f(1)) = (1, 1\cancel{\ln 1}^{\,0} - 1) = (1,-1)$ is a loc min.

EXAMPLE 9 Revenue Analysis The graph of the total revenue $R(x)$ (in dollars) from the sale of x bookcases is shown in Figure 14.

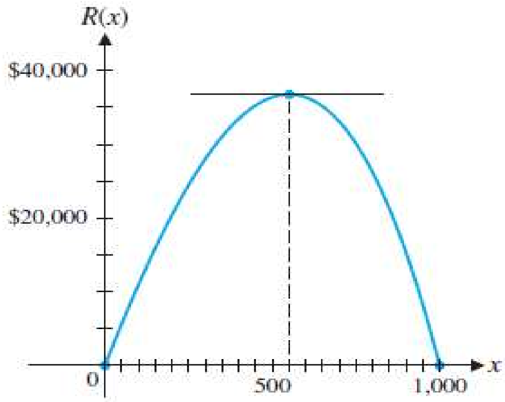


Figure 14 **Revenue**

(A) Write a brief description of the graph of the marginal revenue function $y = R'(x)$, including a discussion of any x intercepts.

(B) Sketch a possible graph of $y = R'(x)$.