Review (Differentiation Rules)

- $$\frac{d}{dx} f^{n}(x) = n f^{n-1}(x) \cdot f'(x)$$

  n ← reel number

  f(x) ← function of x

  - $f(x) = x$; $\frac{d}{dx} x^n = n x^{n-1}$

- $$\frac{d}{dx} a^{f(x)} = a^{f(x)} \cdot \ln a \cdot f'(x)$$

  f(x) ← function

  a ← positive real number

  - $f(x) = x$; $\frac{d}{dx} a^x = a^x \cdot \ln a$
  - $a = e$; $\frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot f'(x)$
  - $a = e$ and $f(x) = x$; $\frac{d}{dx} e^x = e^x$

- $$\frac{d}{dx} \log_b f(x) = \frac{f'(x)}{f(x)} \cdot \frac{1}{\ln b}$$

  - $b = e$; $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$
  - $b = e$ and $f(x) = x$; $\frac{d}{dx} \ln x = \frac{1}{x}$

- $\frac{d}{dx}(f \cdot g) = f'g + g'f$ "Product Rule"

  $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$ "Qutient Rule"

  $\frac{d}{dx}(f \circ g)(x) = \frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$

Implicit Differentiation :

Consider the eqn :

$$3x^2 + y - 2 = 0 \quad \cdots \quad (1)$$

implicitly

or

$$F(x,y) = 3x^2 + y - 2 = 0 .$$

Can you solve eqn(1) for y in terms of x ?

Yes .

excplicitly

$$y = 2 - 3x^2 \quad -- \quad (2)$$

and we may write eqn (2) as

$y = f(x)$ where $f(x) = 2 - 3x^2$.

We say that eqn (2) defines function $y = f(x)$ explicitly and eqn(1) defines y implicitly .

Does every func. can be expressed explicitly ?

Sometimes, it is either difficult or impossible to solve

$$F(x,y) = 0$$

for y explicitly in terms of x.

e.g. 1) $x^2 y^5 - 3xy + 5 = 0$

2) $e^y - y + 3x = 0$

Question. How can we take the derivative of an implicitly written function; $F(x,y)=0$?

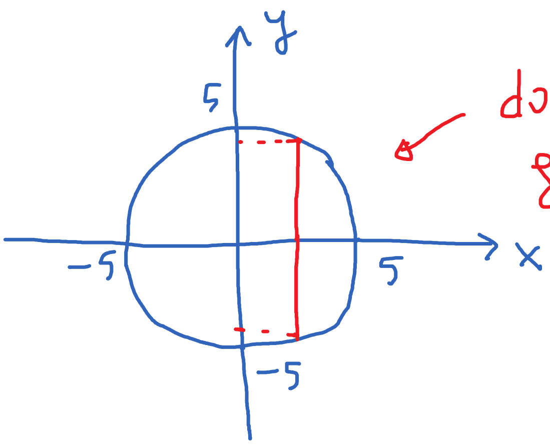
We can use implicit diff.

- differentiate both sides of the eqn wrt x, and (don't forget that $y=y(x)$),
- then solve for y' $\left(\frac{dy}{dx}\right)$

One more reason for applying implicit diff:
Consider the eqn:

$$x^2+y^2 = 25$$

This eqn describes a circle with radius 5



does not represent graph of a func.

This is not a func and we cannot write it as
$y = \cdots$ (unless we split the circle in upper and lower half)

So, how we compute the slope of points on this curve?

Ex. Given $x^2+y^2-25=0$ Find $y' = \frac{dy}{dx}$ and the slope of the graph at $x=3$

Soln.

$$x^2+y^2-25=0 \quad , \; y=y(x)$$

$$\frac{d}{dx}\left(x^2+y^2-25\right) = \frac{d}{dx}(0)$$

$$\boxed{\frac{d}{dx} f^2(x) = 2f(x)\, f'(x)}$$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 - \frac{d}{dx}25 = \frac{d}{dx}0$$

$$2x + 2y\frac{dy}{dx} - 0 = 0$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

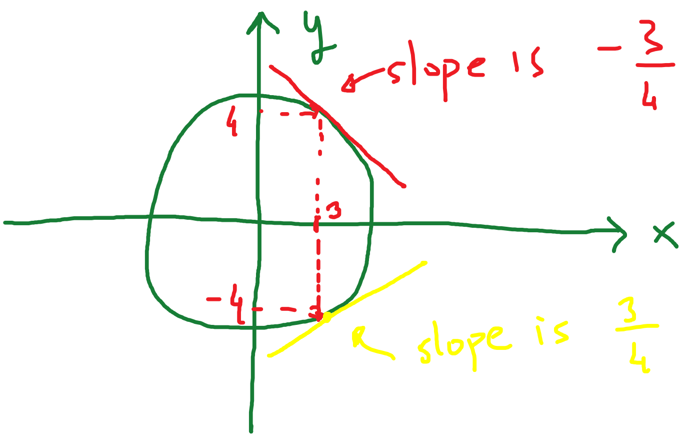
Slope at $x=3$?

$$3^2+y^2 = 25 \Rightarrow y^2 = 25-9 = 16$$

$$\Rightarrow y = \mp 4$$

The points are. $(3,-4)$, $(3,4)$

slopes are ; $\frac{dy}{dx} = -\frac{3}{-4} = \frac{3}{4}$, $\frac{dy}{dx} = -\frac{3}{4}$



Ex Let $y - xy^2 + x^2 = -1$. Find y' at $x=1$

Soln. Take derivative of both sides wrt x:

$$\frac{d}{dx}y - \frac{d}{dx}(xy^2) + \frac{d}{dx}(x^2) = \frac{d}{dx}(-1)$$

$$y' - (1\,y^2 + x\,2y\,y') + 2x = 0$$

$$y' - y^2 - 2xy\,y' + 2x = 0$$

$$y' - 2xy\,y' = y^2 - 2x$$

$$y'(1-2xy) = y^2 - 2x$$

$$y' = \frac{y^2 - 2x}{1-2xy}$$

$x = 1 \Rightarrow y - y^2 + 1^2 + 1 = 0 \Rightarrow -y^2 + y + 2 = 0$

$\Rightarrow y^2 - y - 2 = 0$

$\Rightarrow (y-2)(y+1) = 0$

$\Rightarrow y = 2,\ y = -1$

At $\overset{x}{(1},\ \overset{y}{2)}$: $\frac{dy}{dx} = \frac{2^2 - 2}{1-2.1.2} = \frac{2}{-3}$

At $(1,-1)$. $\frac{dy}{dx} = \frac{(-1)^2 - 2.1}{1-2.1.(-1)} = \frac{-1}{3}$ //

Ex. Find x' for $x=x(t)$ defined implicitly by

$$t\ln x = xe^t - 1$$

and evaluate $x' = \frac{dx}{dt}$ at $(t,x)=(0,1)$.

Soln. Don't forget that x is func of t. $x=x(t)$

Take derivative of both sides wrt t.

$$\frac{d}{dt}(t\ln x) = \frac{d}{dt}(xe^t - 1)$$

$$1\cdot\ln x + t\frac{d}{dt}\ln x = \frac{dx}{dt}e^t + x\frac{d}{dt}e^t - \underbrace{\cancel{\frac{d1}{dt}}}_{0}$$

$$\ln x + t\frac{x'}{x} = x'e^t + xe^t$$

Solve for x':

$$\frac{t}{x}x' - e^t x' = -\ln x + xe^t$$

$$x'\left(\frac{t}{x} - e^t\right) = xe^t - \ln x$$

$$x' = \frac{xe^t - \ln x}{t/x - e^t}$$

$$x'\Big|_{(0,1)}^{t,x} = \frac{1\cdot \cancel{e^0}^{1} - \cancel{\ln 1}^{0}}{\frac{0}{1} - \cancel{e^0}^{1}} = \frac{1}{-1} = -1$$ //