Second Derivatives and Graphs

Let $x \in (0,\infty)$.

i) $f(x) = x^2$

$f'(x) = 2x$

$f'(x) = 2x > 0$ for $x>0$.

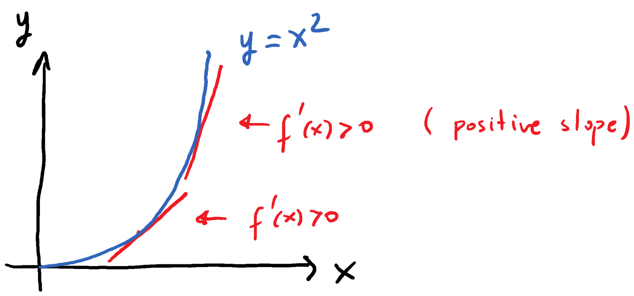
f is increasing on $(0,\infty)$

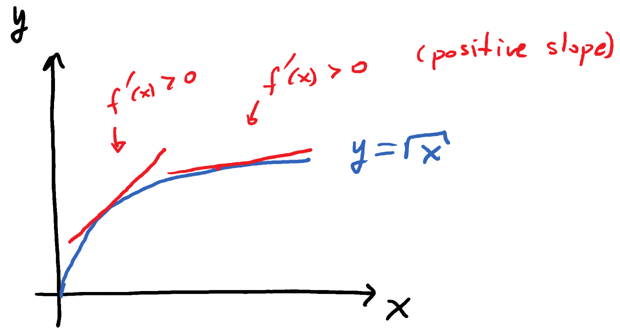
ii. $f(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

$f'(x) > 0$ for all $x>0$.

f is increasing on $(0,\infty)$.





Here, f is increasing with an increasing rate.

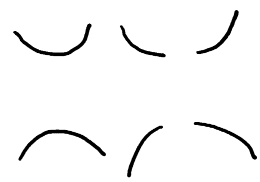
Slope of tangent lines are increasing. Namely, $f'(x)$ is increasing. Hence $(f')' = f''(x)$ is positive.

Here, f is increasing with a decreasing rate.

Slope of tangent lines are decreasing. Namely, $f'(x)$ is decreasing. So, $(f')' = f''(x)$ is negative.

Defn (Concavity) The graph of a func. f is concave upward (C.U.) on (a,b) if $f''(x) > 0$ on (a,b) and is concave downward (C D) on (a,b) if $f''(x) < 0$ on (a,b)

f''	f'	Graph of f	Examples
$+$	$\nearrow$	C U	
$-$	$\searrow$	C D	



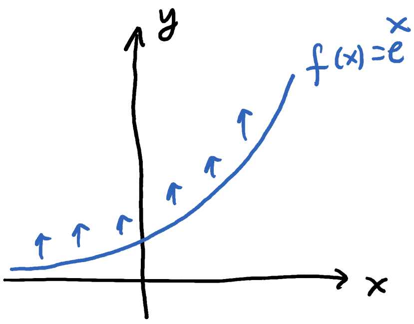
Ex. Determine the intervals of concavity of each function

a) $f(x) = e^x$ b) $g(x) = \ln x$ c) $h(x) = x^3$

Soln. a) $f'(x) = e^x \Rightarrow f''(x) = e^x$

$f''(x) = e^x > 0$ for all $x \in \mathbb{R}$. Hence

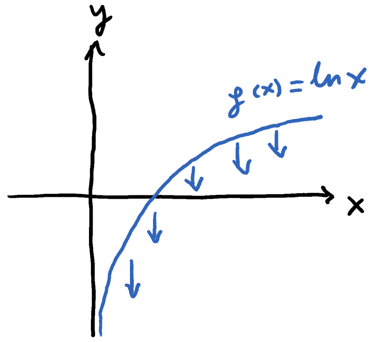
f is CU for all $x \in \mathbb{R}$.



b) $g(x) = \ln x$; $g'(x) = \frac{1}{x} \Rightarrow g''(x) = -\frac{1}{x^2}$

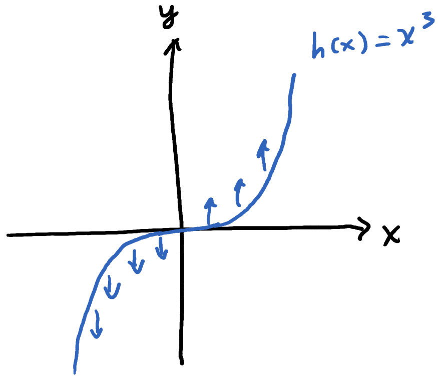
First of all $g(x) = \ln x$ is defined for $x > 0$. $g''(x) < 0$ for all $x \in (0, \infty)$.

Hence, g is CD on $(0, \infty)$



c) $h(x) = x^3$; $h'(x) = 3x^2 \Rightarrow h''(x) = 6x$

For $x > 0$ $h''(x) = 6x > 0$ and for $x < 0$ $h''(x) = 6x < 0$. Hence h is CU on $(0, \infty)$ and CD on $(-\infty, 0)$.



x	$-\infty$	0	∞
h''	$-$	0	$+$
h	$\frown$		$\smile$

Ex. Determine the intervals of concavity of each function

a) $f(x) = -e^{-x}$ b) $g(x) = \ln \frac{1}{x}$ c) $h(x) = x^{1/3}$

Solve it!

Inflection Point.

An inflection point is a point on the graph of func. where the concavity changes.

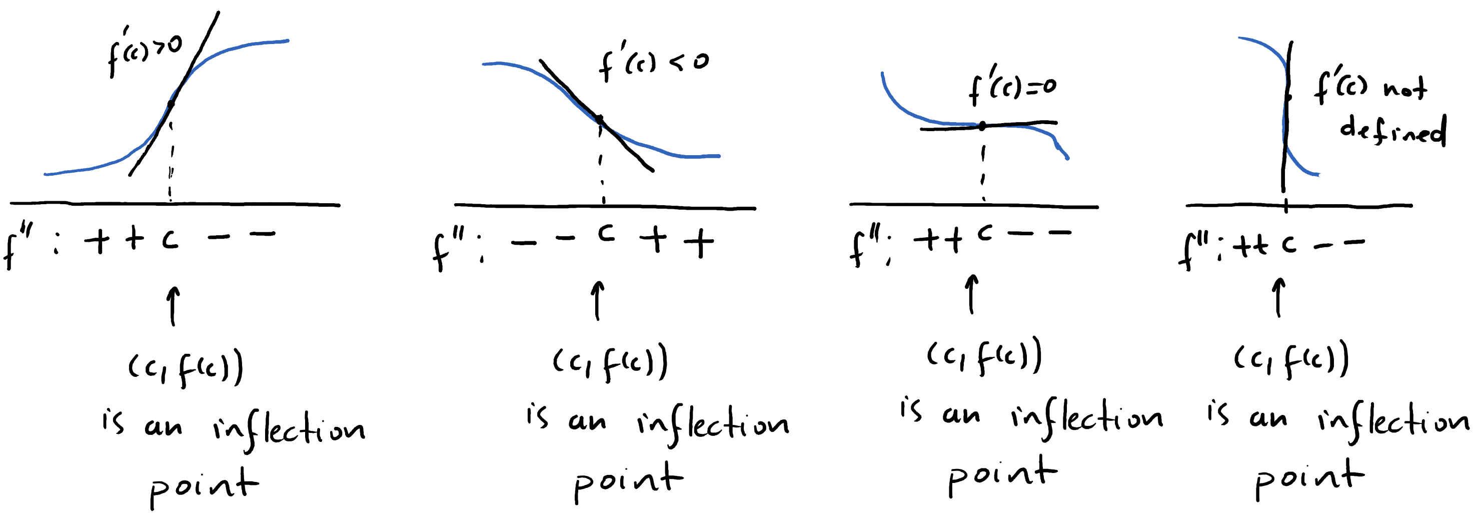
Teorem. If $(c, f(c))$ is an inflection point of f, then either $f''(c) = 0$ or $f''(c)$ does not exist.

Remark. The points that makes $f''(x) = 0$ or $f''(x)$ d.n.e are candidates for being an inflection point.

To find inflection point:

1) Find the numbers c such that $f''(c) = 0$ or $f''(c)$ d.n.e.
2) Does f'' change sign at c?
3) Is c in the domain of f?

If all answers are yes, then $(c, f(c))$ is an inf. point of f.



Ex. Find inflection points of

a) $f(x) = x^3 - 6x^2 + 9x + 1$

b) $f(x) = \ln(x^2 - 4x + 5)$

Soln. a) $f'(x) = 3x^2 - 12x + 9$

$f''(x) = 6x - 12 = 0 \Rightarrow 6(x-2) = 0$

$\Rightarrow x = 2$ candidate

x	$-\infty$	2	∞
f''	$-$	0	$+$
f	$\frown$		$\smile$

$(2, f(2))$

f is CD on $(-\infty, 2)$ and CU on $(2, \infty)$. Hence $(2, f(2))$ is an inflection point.

b) What is domain of $f(x) = \ln(x^2 - 4x + 5)$?

$x^2 - 4x + 5 = (x-2)^2 + 1 > 0$ for all $x \in \mathbb{R}$.

Hence f is defined everywhere.

$$f'(x) = \frac{2x-4}{x^2-4x+5} \quad ; \quad f''(x) = \frac{2(x^2-4x+5) - (2x-4)(2x-4)}{(x^2-4x+5)^2}$$

$$= \frac{-2(x-1)(x-3)}{(x^2-4x+5)^2}$$

$f''(x) = 0 \Rightarrow x = 1$ or $x = 3$.

x	$-\infty$	1		3	∞
f''	$-$	0	$+$	0	$-$
f	$\frown$		$\smile$		$\frown$

inf. points

f is CD on $(-\infty, 1)$ and $(3, \infty)$, CU on $(1, 3)$. $(1, f(1))$ and $(3, f(3))$ are inflection points of f.

Exercise. Find inflection points of

a) $f(x) = x^3 - 9x^2 + 24x - 10$ b) $f(x) = \ln(x^2 - 2x + 5)$

Point of Diminishing Returns (Inflection Points)

If a company decides to increase spending on advertising, it would expect sales to increase. At first, sales will increase at an increasing rate and then increase at a decreasing rate. The dollar amount x at which the rate of change of sales goes from increasing to decreasing is called the **point of diminishing returns**. This is also the amount at which the rate of change has a maximum value. Money spent beyond this amount may increase sales but at a lower rate.

EXAMPLE 7 Maximum Rate of Change Currently, a discount appliance store is selling 200 large-screen TVs monthly. If the store invests $x thousand in an advertising campaign, the ad company estimates that monthly sales will be given by

N'

$$\underline{N(x) = 3x^3 - 0.25x^4 + 200 \qquad \boxed{0 \le x \le 9}}$$

When is the rate of change of sales increasing and when is it decreasing? What is the point of diminishing returns and the maximum rate of change of sales? Graph N and N' on the same coordinate system.

Inflection Point of N'

Max $N' = ?$

Matched Problem 7 Repeat Example 7 for

$$N(x) = 4x^3 - 0.25x^4 + 500 \qquad 0 \le x \le 12$$

Soln. $N'(x) = 9x^2 - x^3$; $N''(x) = 18x - 3x^2 = 3x(6-x)$

a) When $\underbrace{N'(x) \nearrow}_{(N')' > 0}$, when $\underbrace{N'(x) \searrow}_{(N')' < 0}$?

$N''(x) = 0 \Rightarrow 3x(6-x) = 0 \Rightarrow x = 0,\ x = 6$

x	0		6		9
N''		+	0	−	
N'		↗		↘	

$N'(6) = 108$

loc max

rate of change of sales

N' is increasing on $(0,6)$ and decreasing on $(6,9)$

The max rate of change is $N'(6) = 108$

x	0		6		9
N''		+	0	−	
N		∪		∩	

Inflection Point

The point of diminishing return is $(6, N(6)) = (6, 524)$.

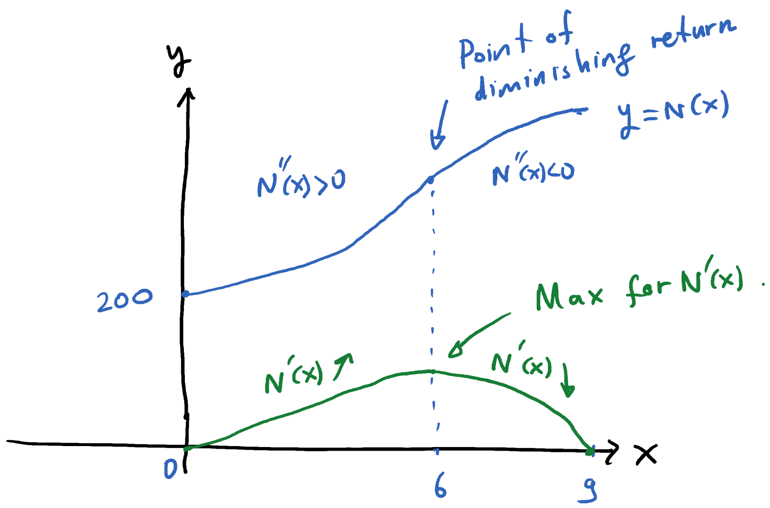
So if the store spends \$6 000 on advertising, monthly save are expected to be $N(6)=524$ TVs, and sales are expected to increase at a rate of $N'(6)=108$ TVs per thousand dollars spent on advertising. Money spent beyond the \$6000 would increase sales, but a lower rate.

x	0	6	9
N''	+	0	−
N'	↗		↘
N	∪		∩

$$N(x) = 3x^3 - 0.25x^4 + 200$$

$$N'(x) = 9x^2 - x^3$$

$$N'(0)=0, \quad N'(9)=0$$



87. Revenue. The marketing research department of a computer company used a large city to test market the firm's new laptop. The department found that the relationship between price p (dollars per unit) and the demand x (units per week) was given approximately by

$$p = 1{,}296 - 0.12x^2 \qquad 0 < x < 80$$

So, weekly revenue can be approximated by

$$R(x) = xp = 1{,}296x - 0.12x^3 \qquad 0 < x < 80$$

(A) Find the local extrema for the revenue function.

(B) On which intervals is the graph of the revenue function concave upward? Concave downward?

Soln. a) Find the c.p. of R:

$R'(x) = 1296 - 0.36x^2$

$R'(x) = 0 \Rightarrow 0.36x^2 = 1296$

$\Rightarrow x^2 = \dfrac{1296}{0.36} = 3600$

$\Rightarrow x = \mp 60$

$\Rightarrow x = 60$ is the only c.p. of R.

x	0		60		80
R'		+	0	−	
R		↗		↘	

$R(60)$ is loc max.

R is increasing on $(0,60)$ and decreasing on $(60,80)$.

b) $R''(x) = -0.72x$

$R''(x) = 0 \Rightarrow x = 0 \notin (0,80)$

x	0		80
R''		−	
R		∩	

R is concave down on $(0,80)$.

88. Profit. Suppose that the cost equation for the company in Problem 87 is

$$C(x) = 830 + 396x$$

(A) Find the local extrema for the profit function.

(B) On which intervals is the graph of the profit function concave upward? Concave downward?

Soln.

$P(x) = R(x) - C(x)$

$= 900x - 0.12x^3 - 830, \quad 0 < x < 80$

a) $P'(x) = 900 - 0.36x^2$

$P'(x) = 0 \Rightarrow 0.36x^2 = 900$

$\Rightarrow x^2 = \dfrac{900}{0.36} = 2500$

$\Rightarrow x = \mp 50$

$\Rightarrow x = 50$

$x = 50$ is the only critical point (c.p.) of $P(x)$.

x	0		50		80
P'		+	0	−	
P		↗		↘	

$P(50)$ is loc. max.

P is increasing on $(0,50)$ and decreasing on $(50\ 80)$.

b) $P''(x) = -0.72x$

So $P(x)$ is always concave downward on $(0,80)$, since $P''(x) < 0$ for all $x \in (0,80)$.

91. Production: point of diminishing returns. A T-shirt manufacturer is planning to expand its workforce. It estimates that the number of T-shirts produced by hiring x new workers is given by

$$T(x) = -0.25x^4 + 5x^3 \qquad 0 \le x \le 15$$

When is the rate of change of T-shirt production increasing and when is it decreasing? What is the point of diminishing returns and the maximum rate of change of T-shirt production? Graph T and T' on the same coordinate system.

Soln. Homework.

94. Advertising: point of diminishing returns. A company estimates that it will sell $N(x)$ units of a product after spending $\$x$ thousand on advertising, as given by

$$N(x) = -0.25x^4 + 13x^3 - 180x^2 + 10{,}000 \qquad 15 \le x \le 24$$

When is the rate of change of sales increasing and when is it decreasing? What is the point of diminishing returns and the maximum rate of change of sales? Graph N and N' on the same coordinate system.

Soln. $N'(x) = -x^3 + 39x^2 - 360x$

$$\begin{aligned} N''(x) &= -3x^2 + 78x - 360 \\ &= -3(x^2 - 26x + 120) \\ &= -3(x-20)(x-6) \end{aligned}$$

$N''(x) = 0 \Rightarrow x = 20$ and $\cancel{x = 6}$

not in the domain

x	15		20		24
N''		$+$	0	$-$	
N'		$\nearrow$		$\searrow$	
N		$\cup$		$\cap$	

N' is increasing on $(15, 20)$ and decreasing on $(20, 24)$. Hence $N'(20) = 400$ is the maximum rate of change of sales.

On the other hand, $N(20) = 2000$ is the point of diminishing return.

$N(15) = 718.75$, $N(24) = 3088$

$N'(15) = N'(24) = 0$

$N'(20)$ is the loc. max