Faculty of Engineering
Mathematical Analysis I
Fall 2018
Exercises 2: *Limit-Continuity*

1. Evaluate the following limits or explain why they do not exist (do not use l'Hospital's Rule).

   (a) $\lim_{x\to 3} \dfrac{x^2-9}{|x-3|}$

   (b) $\lim_{x\to 3} \dfrac{x^3-27}{x^2-9}$

   (c) $\lim_{x\to\infty} \left(\sqrt{9x^2+3}-\sqrt{x^2-x+1}\right)$

   (d) $\lim_{x\to 1} \dfrac{\sqrt[3]{x}-1}{\sqrt{x}-1}$

   (e) $\lim_{x\to 2} \dfrac{2^x+2^{3-x}-6}{\sqrt{2^{-x}}-2^{1-x}}$

   (f) $\lim_{x\to 1} \dfrac{x-1}{\sqrt[3]{x-1}-\sqrt[3]{1-x}}$

   (g) $\lim_{x\to 1} \dfrac{\sqrt{x}+\sqrt{x-1}-1}{\sqrt{x^2-1}}$

   (h) $\lim_{x\to 0} x^2 \cos\frac{3}{x}$

   (i) $\lim_{x\to 0} \dfrac{\sin 2x}{4x^2-x}$

   (j) $\lim_{x\to -\infty} \dfrac{\sqrt{x^2+3x+5}+x}{x+\sqrt{x^2-x+1}}$

   (k) $\lim_{x\to 0} \dfrac{\sin(16x)}{x+1-\cos x}$

   (l) $\lim_{x\to\infty} \dfrac{\sin x}{e^x}$

   (m) $\lim_{x\to 2^+} \dfrac{\sqrt{x-2}-\sqrt{2}+\sqrt{x}}{\sqrt{x^2-4}}$

   (n) $\lim_{x\to 0} \dfrac{\sin 3x}{\sin 7x}$

   (o) $\lim_{x\to 0} \sqrt{x^3+x^2}\cos\left(\frac{\pi}{x}\right)$

2. Find numbers a and b such that $\lim_{x\to 0} \dfrac{\sqrt{ax+b}-2}{x} = 1$.

3. If $\lim_{x\to a}\left[f(x)+g(x)\right]=3$ and $\lim_{x\to a}\left[f(x)-g(x)\right]=2$, find $\lim_{x\to a} f(x)\,g(x)$.

4. Consider the functions

$$f(x) = \begin{cases} \frac{\sin 2x}{2x}, & x<0, \\ 1, & x=0, \\ \frac{\cos x}{x^3+1}, & x>0 \end{cases}$$

   and

$$g(x) = \begin{cases} \frac{|x|}{x}, & x\neq 0, \\ 1, & x=0 \end{cases}.$$

   Are the functions f and g continuous at $x=0$? Explain your answer.

5. Find the constants m and n so that the following functions are continuous.

   (a) $f(x) = \begin{cases} \frac{x^2-4}{x-2}, & x\neq 2, \\ m, & x=2 \end{cases}$

   (b) $f(x) = \begin{cases} mx-n, & x<1, \\ 5, & x=1, \\ 2mx+n, & x>1 \end{cases}$

   (c) $f(x) = \begin{cases} \frac{\sin^2 x}{x^2-x}, & x\neq 0, \\ m, & x=0 \end{cases}$

6. Show that there is a root of the given equation in the specified interval.

   (a) $e^{-x} = \ln x$, $(1,2)$
   (b) $\cos x = x$, $(0,1)$

7. If $f(x) = x^6 + 2x - 7$, show that there is a number c such that $f(c) = 25$.

8. Is there a number that is exactly 1 more than its cube?

9. Use the Intermediate Value Theorem to prove that there is a positive c such that $c^2 = 2$.

10. The gravitational forced exerted by Earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \dfrac{GMr}{R^3}, & \text{if} \quad r < R, \\ \dfrac{GM}{r^2}, & \text{if} \quad r \geq R \end{cases}$$

where M is the mass of Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r?

11. Let

$$f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1, \\ 0, & x = 1 \end{cases}$$

and

$$g(x) = \begin{cases} 1 + x\sin\left(\frac{121}{x}\right), & x \neq 0, \\ 1, & x = 0 \end{cases}$$

are given.

   (a) Evaluate $\lim_{x\to 1} f(x)$.
   (b) Evaluate $\lim_{x\to 0} g(x)$.
   (c) Evaluate $\lim_{x\to 0} (f \circ g)(x)$.

12. Determine whether the statement is true or false. If it is true, explain why. If it is false, give an example that disproves the statement.

   (a) If $\lim_{x\to a} f(x) = 3$ and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ does not exist.
   (b) If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ does not exist.
   (c) If $\lim_{x\to a} f(x) g(x)$ exists, then the limit must be $f(a) g(a)$.
   (d) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a} f(x) g(x)$ does not exist.
   (e) If the line $x = 2$ is a vertical asymptote of $y = f(x)$, then f is not defined at 2.
   (f) If $f(2) > 0$ and $f(4) < 0$, then there exists a number c between 2 and 4 such that $f(c) = 0$.
   (g) If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} [f(x) - g(x)] = 0$.
   (h) If f is not continuous at 5, then $f(5)$ is not defined.
   (i) A function may has infinitely many vertical asymptote.
   (j) If $f(x) > 5$ for all x and $\lim_{x\to 0} f(x)$ exists, then $\lim_{x\to 0} f(x) > 5$.
   (k) If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ are exist, then $\lim_{x\to a} f(x)$ exists.
   (l) A function may has at most two horizontal asymptote.
   (m) If the line $y = 2$ is a horizontal asymptote of $y = f(x)$, then this line does not cross the graph of $y = f(x)$.