Faculty of Engineering
Mathematical Analysis I
Fall 2018
Exercises 4
Applications of Differentiation:
Related Rates, L'Hospital Rule, Curve Sketching, Optimization Problems

1. Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

2. A spherical balloon is inflated with helium at the rate of 100π ft/min^3. How fast is the balloon's radius increasing at the instant the radius is 5 ft? How fast is the surface area increasing?

3. A 13-ft ladder is leading against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec. How fast is the top of the ladder sliding down the wall then?

4. A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle moving at a constant rate of 17 ft/sec passes under it. How fast is the distance between the bicycle and balloon increasing 3 sec later?

5. The volume of a cube is increasing at a rate of 10 cm^3/min. How fast is the surface area increasing when the length of an edge is 30 cm?

6. One train travels west toward Denver at 120 mph, while a second train 90 mph travels north away from Denver at 90 mph. At time $t = 0$, the first train is 10 miles east and the second train 20 miles north of the Denver station. Calculate the rate at which the distance between the trains is changing:

   (a) At time $t = 0$
   (b) 10 min later

7. Evaluate the following limits (You can use L'Hospital rule).

   (a) $\lim_{x\to 0^+} x^x$
   (b) $\lim_{x\to 0} \dfrac{\sin x}{e^x - e^{-x}}$
   (c) $\lim_{x\to \frac{\pi}{2}^+} \dfrac{\tan x}{\tan 3x}$
   (d) $\lim_{x\to 0^+} x^{1/\ln x}$
   (e) $\lim_{x\to \infty} x \sin \frac{1}{x}$
   (f) $\lim_{x\to \infty} \left(1 - \dfrac{3}{x}\right)^{2x}$
   (g) $\lim_{x\to 0^+} \left(\dfrac{3x+1}{\sin x} - \dfrac{1}{x}\right)$

8. A rancher wishes to build a rectangular corral of 128000 ft^2 with one side along a vertical cliff. The fencing along the cliff cost \$1.50 per foot, whereas along the other three sides the fencing cost \$2.50 per foot. Find the dimensions of the corral so that the cost of fencing is minimum.

9. An open-top box is to be made by cutting small congruent squares from the corners of a 12×12 m^2 sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

10. A piece of wire of length L is bent into the shape of a rectangle. Which dimensions produce the rectangle of maximum area?

11. A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

12. A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?

13. What are the dimensions of the lightest opentop right circular cylindrical can that will hold a volume of 1000 cm^3?

14. The U.S. Postal Service will accept a box for domestic shipment only if the sum of its length and girth (distance around) does not exceed 108 inch. What dimensions will give a box with a square base the largest possible volume?

15. A rectangular storage container with an open top is to have a volume of 10 m^3 . The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.

16. Let $f(x) = \frac{x^2}{x^2-1}$.

   (a) Find the domain and intercepts of f.
   (b) Find all asymptotes of the graph of f.
   (c) Find intervals of increase-decrease, the critical points and local extrema of f.
   (d) Find the intervals of concavity and the inflection points of f.
   (e) Sketch the graph of f.

17. Let $f(x) = x - 1 + \frac{4}{x+3}$.

   (a) Find the domain and intercepts of f.
   (b) Find all asymptotes of the graph of f.
   (c) Find intervals of increase-decrease, the critical points and local extrema of f.
   (d) Find the intervals of concavity and the inflection points of f.
   (e) Sketch the graph of f.

18. Let $f(x) = \frac{e^x}{x}$.

   (a) Find the domain and intercepts of f.
   (b) Find all asymptotes of the graph of f.
   (c) Find intervals of increase-decrease, the critical points and local extrema of f.
   (d) Find the intervals of concavity and the inflection points of f.
   (e) Sketch the graph of f.

19. Let $f(x) = \frac{x^3}{x^3-1}$.

   (a) Find the domain and intercepts of f.
   (b) Find all asymptotes of the graph of f.
   (c) Find intervals of increase-decrease, the critical points and local extrema of f.
   (d) Find the intervals of concavity and the inflection points of f.
   (e) Sketch the graph of f.

20. Let $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$.

   (a) Find the domain and intercepts of f.
   (b) Find all asymptotes of the graph of f.
   (c) Find intervals of increase-decrease, the critical points and local extrema of f.
   (d) Find the intervals of concavity and the inflection points of f.
   (e) Sketch the graph of f.

21. Let $f(x) = \left(x^4+1\right)/x$. Show that

$$\lim_{x\to\pm\infty} \left[f(x) - x^3\right] = 0.$$

   This shows that the graph of f approaches the graph of $y = x^3$, and we say that the curve $y = f(x)$ is asymptotic to the curve $y = x^3$. Use this fact to help sketch the graph of f.

22. True/False

   (a) If f is increasing on an interval, then $f'(x) > 0$ on the interval.
   (b) If $f'(x) > 0$ on an interval, then f is increasing on that interval.
   (c) If $f'(c) = 0$, then f has an extreme value at c.
   (d) If f has a local extreme value at c, and if $f'(c)$ exists, then $f'(c) = 0$.
   (e) If $f''(c) = 0$, then $(c, f(c))$ is a point of inflection.
   (f) If $f''(x) < 0$ for all x in the interval (a, b), then the graph of f is concave down on that interval.
   (g) If $f(c)$ is relative maximum, then $f'(c) = 0$.

(h) A limit of the form 1^{∞} is always 1.

(i) There exists a function f such that $f(x) > 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.

(j) A limit of the form $\frac{\infty}{\infty}$ is always indeterminate.

(k) Every absolute extreme is also a local extrema.

(l) A limit of the form $\infty - \infty$ is always 0.

(m) If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.