(b) $\lim_{x\to 0^+} x^x$

$$y = x^x \Rightarrow \ln y = x\ln x \Rightarrow y = e^{x\ln x} \quad 02$$

$$\lim_{x\to 0^+} y = e^{\lim_{x\to 0^+} x\ln x} \quad 02$$

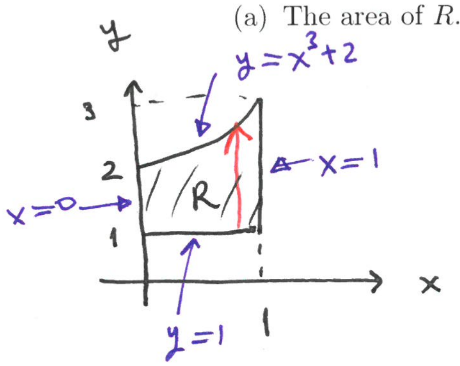
$$\lim_{x\to 0^+} x\ln x \overset{0\cdot\infty}{=} \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{\frac{\infty}{\infty}}{\underset{LH}{=}} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \quad 04$$

$$= \lim_{x\to 0^+} (-x) = 0$$

$$\therefore \lim_{x\to 0^+} x^x = e^0 = 1. \quad 02$$

(30 point) 6. Let R be the region bounded by the curves $y = x^3 + 2$, $y = 1$, $x = 0$ and $x = 1$. Set up, **but do not evaluate**, an integral for;

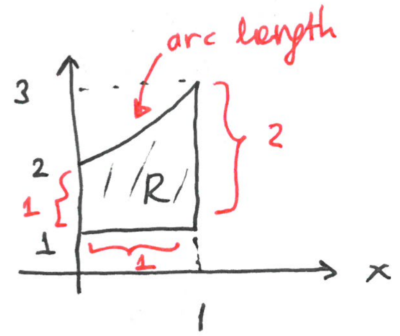
(a) The area of R.



$$A(R) = \int_0^1 \left[(x^3+2) - 1\right] dx$$

03 07

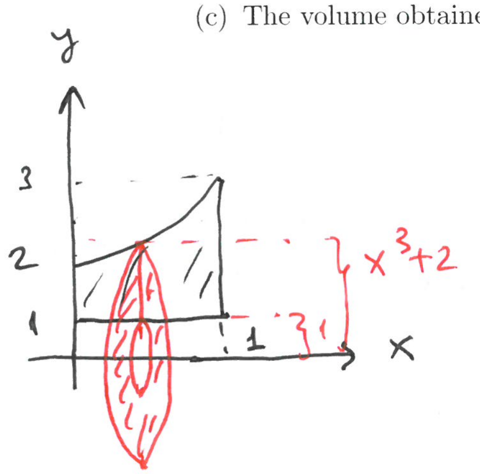
(b) The perimeter of R.

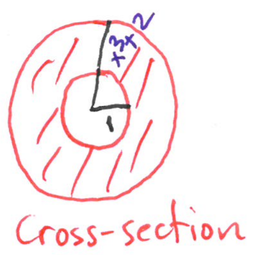


$$P(R) = 1 + 1 + 2 + \int_0^1 \sqrt{1 + \left[\frac{d}{dx}(x^3+2)\right]^2}\, dx \quad 07$$

$$= 4 + \underbrace{\int_0^1 \sqrt{1 + 9x^4}\, dx}_{\text{arc length}} \quad 03$$

(c) The volume obtained by rotating R about the x-axis.





$$A(x) = \pi (x^3+2)^2 - \pi \cdot 1^2$$

$$V(R) = \int_0^1 A(x)\, dx$$

$$= \pi \int_0^1 \left[(x^3+2)^2 - 1\right] dx$$

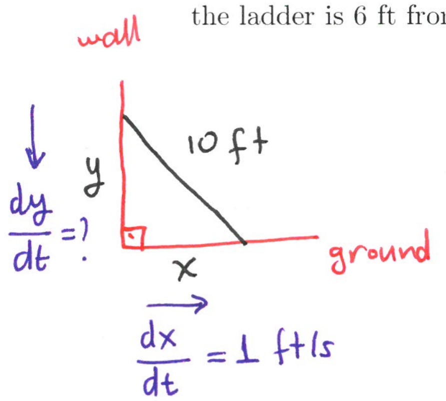
03 07

İstanbul Commerce University
Faculty of Engineering
MAT121-Mathematical Analysis I
Final Exam

Name-Surname:		Abdullah YENER
Student Number:	ANSWER KEY	11.01.2018, 14:00
Department:		C207-C307

Attention. The test duration is 90 minutes. The use of a calculator, cell phone or other equivalent electronic devices or documents are not allowed. Show your work in a reasonable detail. A correct answer without proper or too much reasoning may not get any credit. Good luck.

(15 point) 1. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?



$$\frac{dx}{dt} = 1 \text{ ft/s}, \quad \text{when } x = 6 \text{ ft} \quad \frac{dy}{dt} = ?$$

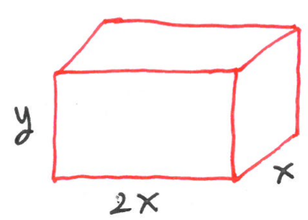
$$x^2 + y^2 = 100 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

When $x = 6$ ft, from the Pythagorean Thm $y = 8$ ft. Hence, we get

$$\frac{dy}{dt} = -\frac{6}{8}\cdot 1 = -\frac{3}{4} \text{ ft/s}.$$

(15 point) 2. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs 10$ per square meter. Material for the sides costs 6$ per square meter. Find the cost of materials for the cheapest such container.



(Volume) $V = 2x\cdot x\cdot y \Rightarrow 2x^2y = 10 \Rightarrow y = \frac{5}{x^2}$

(Cost) $C = 10(2x^2) + 6(6xy) = 20x^2 + 36xy$

Since $y = \frac{5}{x^2}$, we have $C(x) = 20x^2 + \frac{180}{x}$.

$$C'(x) = 40x - \frac{180}{x^2}; \quad C'(x) = 0 \Rightarrow 40x = \frac{180}{x^2}$$

$$\Rightarrow x^3 = \frac{9}{2}$$

$$\Rightarrow x = \sqrt[3]{9/2} \quad \text{critical point}$$

Therefore, the minimum cost is

$$C\left(\sqrt[3]{\tfrac{9}{2}}\right) = 20\left(\sqrt[3]{\tfrac{9}{2}}\right)^2 + \frac{180}{\sqrt[3]{\tfrac{9}{2}}} \ \$.$$

(30 point) 3. Let $f(x) = \frac{x^2}{x^2-1}$.

(a) Find the domain and intercepts of f.

$\mathrm{Dom}\, f = \mathbb{R} \setminus \{-1, 1\}$ 03

$x=0 \Rightarrow y=0$; $(0,0)$ is the intercept of f. 02

(b) Find all asymptotes of the graph of f.

Vertical asymptotes;

02 $\lim_{x \to -1^{\mp}} \frac{x^2}{x^2-1} = \mp\infty \Rightarrow x=-1$ V.A.

02 $\lim_{x \to 1^{\mp}} \frac{x^2}{x^2-1} = \mp\infty \Rightarrow x=1$ V.A.

Horizontal asymptotes;

02 $\lim_{x \to \mp\infty} \frac{x}{x^2-1} = 1 \in \mathbb{R} \Rightarrow y=1$ H.A.

(c) Find intervals of increase and decrease, and the local extrema of f.

$$f'(x) = \frac{2x(x^2-1) - 2x \cdot x^2}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2}$$ 02

x	-1	0	1	
f'	+	+	−	−
f	↗	↗	↘	↘

02 f is increasing on $(-\infty,-1) \cup (-1,0)$
f is decreasing on $(0,1) \cup (1,\infty)$

02 $x=0$ is a local maximum point

(d) Find the intervals of concavity and the inflection points of f.

$$f''(x) = \frac{-2(x^2-1)^2 + 2x\, 2(x^2-1)\, 2x}{(x^2-1)^4} = \frac{6x^2+2}{(x^2-1)^3}$$ 02

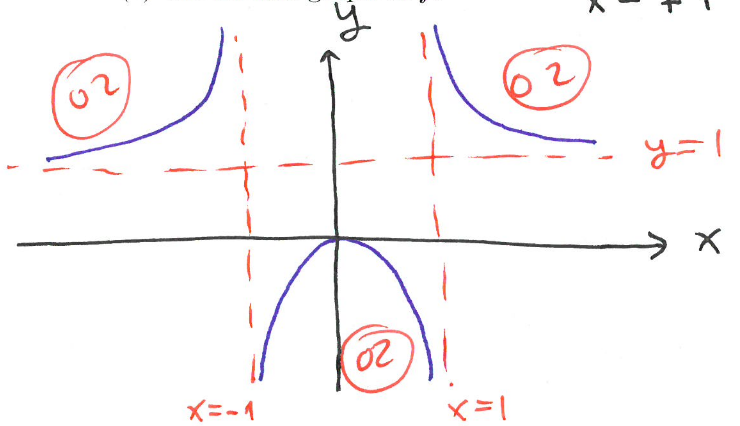
$f''(x) = 0$ No soln.

x	-1	1	
f''	+	−	+
f	∪	∩	∪

02 • f is concave up on $(-\infty,-1) \cup (1,\infty)$
• f is concave down on $(-1,1)$

02 No inflection points, since $x = \mp 1 \notin \mathrm{Dom}\, f$.

(e) Sketch the graph of f.



(30 point) 4. Evaluate the following integrals.

(a) $\int x \sin x\, dx = uv - \int v\, du$ (integration by parts)

$u = x \Rightarrow du = dx$ 05
$dv = \sin x\, dx \Rightarrow v = \int \sin x\, dx = -\cos x$

$\therefore \int x \sin x\, dx = -x\cos x + \int \cos x\, dx$ 05
$= -x\cos x + \sin x + C$

(b) $\int_1^e \frac{dx}{x\sqrt{\ln x}} = \int_0^1 \frac{du}{\sqrt{u}} = \int_0^1 u^{-1/2}\, du = 2\sqrt{u}\,\Big|_0^1 = 2$ 05

$u = \ln x \Rightarrow du = \frac{dx}{x}$
$x=1 \Rightarrow u = \ln 1 = 0$ 05
$x=e \Rightarrow u = \ln e = 1$

Substitution

(c) $\int \frac{dx}{x^2-5x+6}$ (Partial Fraction)

$$\frac{1}{x^2-5x+6} = \frac{1}{(x-2)(x-3)} = \frac{A}{\underset{(x-3)}{x-2}} + \frac{B}{\underset{(x-2)}{x-3}} ; \quad A(x-3)+B(x-2) = 1$$

For $x=3$, $B=1$
For $x=2$, $A=-1$

$$\therefore \int \frac{dx}{x^2-5x+6} = -\int \frac{dx}{x-2} + \int \frac{dx}{x-3}$$ 05

$$= -\ln|x-2| + \ln|x-3| + C = \ln\left|\frac{x-3}{x-2}\right| + C$$ 05

(20 point) 5. Evaluate the following limits (You can use the L'Hospital Rule).

(a) $\lim_{x\to 0} \frac{\int_0^x \cos(t^2)\, dt}{x}$ 05 05

$$\overset{0/0}{\underset{LH}{=}} \lim_{x\to 0} \frac{\frac{d}{dx}\left(\int_0^x \cos t^2\, dt\right)}{\frac{d}{dx}(x)} \underset{\text{F.T.C.}}{=} \lim_{x\to 0} \frac{\cos x^2}{1} = 1.$$