(b) Evaluate $\lim_{x\to\infty}\left(1-\frac{3}{x}\right)^{2x}$ 03

$y=\left(1-\frac{3}{x}\right)^{2x} \Rightarrow \ln y = 2x\ln\left(1-\frac{3}{x}\right) \Rightarrow y = e^{\frac{\ln\left(1-\frac{3}{x}\right)}{1/2x}}$

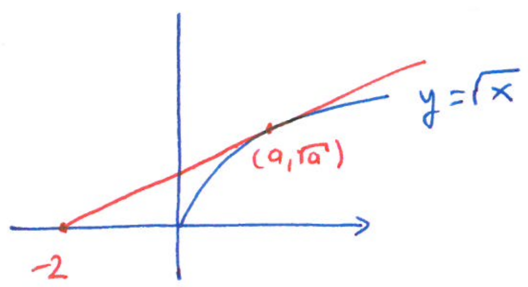
02

$\lim_{x\to\infty} y = e^{\lim_{x\to\infty}\frac{\ln\left(1-\frac{3}{x}\right)}{\frac{1}{2x}}} \overset{LH\ \left(\frac{0}{0}\right)}{=} e^{\lim_{x\to\infty}\frac{\frac{3}{x^2}/\left(1-\frac{3}{x}\right)}{-\frac{1}{2x^2}}}$

e^x is cont.

$= e^{\lim_{x\to\infty}\frac{-6}{1-\frac{3}{x}}} = e^{-6}$ 05

(c) Find an equation of the straight line that passes through the point $(-2,0)$ and is tangent to the curve $y=\sqrt{x}$.



$m_1 = \frac{\sqrt{a}-0}{a-(-2)} = \frac{\sqrt{a}}{a+2}$ 02

$m_2 = \frac{d}{dx}(\sqrt{x})\Big|_{x=a} = \frac{1}{2\sqrt{a}}$ 02

Since m_1 must be equal to m_2, we get

$\frac{\sqrt{a}}{a+2} = \frac{1}{2\sqrt{a}} \Rightarrow 2a = a+2 \Rightarrow a=2$ 02

Hence, eqn of tangent line with slope $\frac{1}{2\sqrt{2}}$ passing through the point $(-2,0)$ is

$y = \frac{1}{2\sqrt{2}}(x+2)$. 04

(10 point) 6. Determine whether the statement is true or false. No justification is needed.

(a) If f is increasing on an interval, then $f'(x)>0$ on the interval. T F

(b) If $f'(x)>0$ on an interval, then f is increasing on that interval. T F

(c) If $f'(c)=0$, then f has an extreme value at c. T F

(d) If f has a local extreme value at c, and if $f'(c)$ exists, then $f'(c)=0$. T F

(e) If $f''(c)=0$, then $(c,f(c))$ is a point of inflection. T F

(f) If $f''(x)<0$ for all x in the interval (a,b), then the graph of f is concave down on that interval. T F

(g) A limit of the form 1^∞ is always 1. T F

(h) If $f(c)$ is relative maximum, then $f'(c)=0$. T F

(i) A limit of the form $\infty-\infty$ is always 0. T F

(j) If f is continuous on a closed interval $[a,b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a,b]$. T F

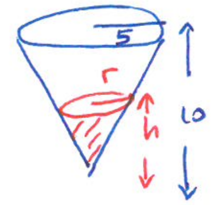
İstanbul Commerce University
Faculty of Engineering
MAT121-Mathematical Analysis I
Final Exam

Name-Surname:
Student Number:
Department:

Abdullah YENER
07.01.2019, 14:00
C-207/307

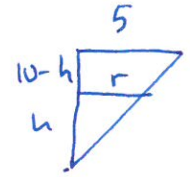
Attention. The test duration is 90 **minutes**. The exam is out of 120 **points**. The use of a calculator, cell phone or other equivalent electronic devices or documents are not allowed. Show your work in a reasonable detail. A correct answer without proper or too much reasoning may not get any credit. Good luck.

(15 point) 1. Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?



Given: $\frac{dV}{dt} = 9\ ft^3/min$ Unknown: $\frac{dh}{dt}$ when $h=6\ ft$

Equation: $V=\frac{1}{3}\pi r^2 h$ 05



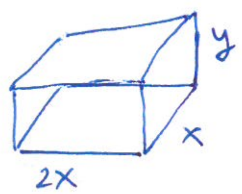
Using similar triangles we get

$\frac{r}{h} = \frac{5}{10} \Rightarrow r=\frac{h}{2} \Rightarrow V=\frac{\pi}{12}h^3$ 05

Now, diff. wrt t: 05

$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt} \Rightarrow \frac{dh}{dt} = \frac{4}{\pi 36}\cdot 9 = \frac{1}{\pi}$ ft/min

(15 point) 2. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.



$V = 2x^2y = 10$

$C(x,y) = 2x^2\cdot 10 + 6xy\cdot 6 = 20x^2+36xy$ 05

$2x^2y=10 \Rightarrow y=\frac{5}{x^2} \Rightarrow C(x) = 20x^2+36x\cdot\frac{5}{x^2}$

$= 20x^2+\frac{180}{x}$ 05

Minimize $C(x)$:

$C'(x) = 40x-\frac{180}{x^2} = 0 \Rightarrow 40x = \frac{180}{x^2} \Rightarrow x^3=\frac{9}{2}$

$\Rightarrow x=\sqrt[3]{9/2}$ 05

$C''(x) = 40+\frac{360}{x^3} \Rightarrow C''(\sqrt[3]{9/2}) = 40+\frac{360\cdot 2}{9} = 120>0$

$\therefore C(\sqrt[3]{9/2})$ is min value and equal to

$C(\sqrt[3]{9/2}) = 20\left(\frac{9}{2}\right)^{\frac{2}{3}} + \frac{180}{\left(\frac{9}{2}\right)^{1/3}}$.

(30 point) 3. Consider $f(x) = \dfrac{x^2-1}{x^2-4}$ with its derivatives: $f'(x) = \dfrac{-6x}{(x^2-4)^2}$ and $f''(x) = \dfrac{6(3x^2+4)}{(x^2-4)^3}$.

(a) Find the domain and intercepts of f.

$\text{Dom} f = \mathbb{R} - \{-2, 2\}$ 02

x-intercepts : $y=0 \Rightarrow x = \mp 1$ 02 y-intercepts: $x=0 \Rightarrow y = \frac{1}{4}$ 02

(b) Find all asymptotes of the graph of f.

- $\displaystyle\lim_{x\to -2^-} \frac{x^2-1}{x^2-4} = \infty$, $\displaystyle\lim_{x\to -2^+} \frac{x^2-1}{x^2-4} = -\infty$ $\therefore$ $x=-2$ is a V.A 02
- $\displaystyle\lim_{x\to 2^-} \frac{x^2-1}{x^2-4} = -\infty$, $\displaystyle\lim_{x\to 2^+} \frac{x^2-1}{x^2-4} = \infty$ $\therefore$ $x=2$ is a V.A. 02
- $\displaystyle\lim_{x\to \mp\infty} \frac{x^2-1}{x^2-4} = 1 \Rightarrow y=1$ is a H.A. 02

(c) Find intervals of increase and decrease, and the local extrema of f if there exists.

04

x		-2		0		2	
f'	$+$		$+$	0	$-$		$-$
f	↗		↗		↘		↘

$(0, f(0)) = (0, \frac{1}{4})$ 02 is a local max.

(d) Find the intervals of concavity and the inflection points of f if there exists.

04

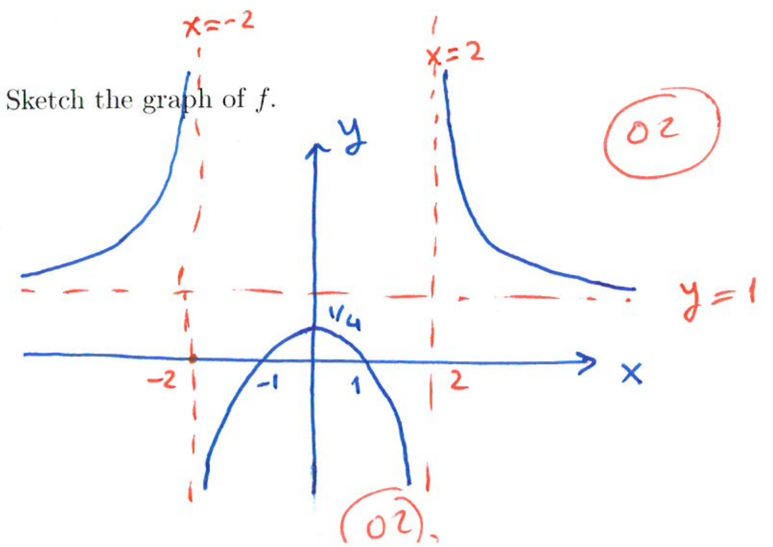
x		-2		2	
f''	$+$		$-$		$+$
f	∪		∩		∪

$f''(x) = 0$ has no soln. 02

Since $-2, 2 \notin \text{Dom} f$, there is no inflection points.

(e) Sketch the graph of f.

02



(20 point) 4. Evaluate the following integrals.

05

(a) $\int x^2 \ln x\,dx = \dfrac{x^3}{3}\ln x - \displaystyle\int \frac{x^3}{3}\,\frac{1}{x}\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$

$$\left(\begin{array}{l} u = \ln x \Rightarrow du = \frac{1}{x}dx \\ dv = x^2 dx \Rightarrow v = \frac{x^3}{3} \end{array} \right)$$

05

"By Integration by Parts Formula"

(b) $\int \cos^3 x \sin^2 x\,dx$

$\cos^2 x + \sin^2 x = 1$ 02

$$= \int (1-\sin^2 x)\sin^2 x \cos x\,dx$$
$$= \int (1-u^2)\,u^2\,du$$
$$= \int u^2 du - \int u^4 du$$
$$= \frac{u^3}{3} - \frac{u^5}{5} + C$$
$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

05

$u = \sin x$

$du = \cos x\,dx$

03

(30 point) 5. This question has three unrelated parts.

(a) Write out the form of the partial fraction decomposition of the function

$$\frac{x^4}{(x^3+x)^2(x^2-x+3)}.$$

$$\frac{x^4}{x^2(x^2+1)^2(x^2-x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2} + \frac{Gx+H}{x^2-x+3}$$

02 02 02 02 02