İstanbul Commerce University
Faculty of Engineering
MAT121-Mathematical Analysis I
Midterm Exam

Name-Surname: Solution Key
ID Number:
Department:

Abdullah YENER
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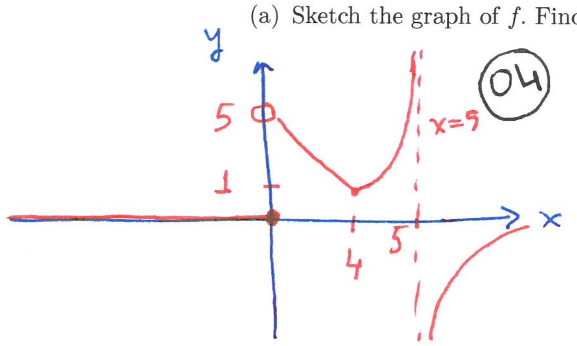
C207-C307

Attention. The test duration is 75 minutes. The exam is out of 110 points. The use of a calculator, cell phone or other equivalent electronic devices or documents are not allowed. Show your work in a reasonable detail. A correct answer without proper or too much reasoning may not get any credit. Good luck.

(8+6+6+6) 1. Let $f(x) = \begin{cases} 0, & \text{if } x \le 0, \\ 5-x, & \text{if } 0 < x < 4 \\ \frac{1}{5-x}, & \text{if } x \ge 4 \end{cases}$.

(a) Sketch the graph of f. Find the domain and range of f.



$\text{Dom} f = \mathbb{R} \setminus \{5\}$ 02

$\text{Range} f = \mathbb{R} \setminus (0,1)$ 02

(b) Does $\lim_{x\to 0} f(x)$ exist? Explain your answer.

02 No. "In the graph there is a break". 04

OR

"$\lim_{x\to 0^-} f(x) = 0 \neq \lim_{x\to 0^+} f(x) = 5$"

(c) Is f continuous at $x = 0$ and at $x = 4$? Explain your answer.

At $x=0$: f is not cont. since $\lim_{x\to 0} f(x)$ d.n.e. 01 02

At $x=4$: f is cont. because no hole or break in the graph of f. 01 02

(d) Is f differentiable at $x = 4$? Explain your answer.

02 No. Since there is a corner in the graph of f. At $x=4$ f has inf. many tangent line. So no differentiability at that point. 04

(8+8+8) 2. Evaluate the following limits or explain why they do not exist (do not use l'Hospital's Rule).

(a) $\lim_{x\to 2} \frac{x^2-4}{|x-2|}$

- $\lim_{x\to 2^-} \frac{x^2-4}{|x-2|} = \lim_{\substack{x\to 2^- \\ (x<2)}} \frac{(\cancel{x-2})(x+2)}{-(\cancel{x-2})} = \lim_{x\to 2^-} -(x+2) = -4$ 03

- $\lim_{x\to 2^+} \frac{x^2-4}{|x-2|} = \lim_{\substack{x\to 2^+ \\ (x>2)}} \frac{(\cancel{x-2})(x+2)}{(\cancel{x-2})} = \lim_{x\to 2^+} x+2 = 4$ 03

Since $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$, limit d.n.e. 02

(b) $\lim_{x\to 0} \sqrt{x^3+x^2}\sin\frac{121}{x}$

$-1 \le \sin\frac{121}{x} \le 1 \quad \forall x \in \mathbb{R}\setminus\{0\}$ 02

Since $\sqrt{x^3+x^2}$ must be positive,

$-\sqrt{x^3+x^2} \le \sqrt{x^3+x^2}\sin\frac{121}{x} \le \sqrt{x^3+x^2}$

holds. By Squeeze Thm,

$\lim_{x\to 0} -\sqrt{x^3+x^2} = \lim_{x\to 0}\sqrt{x^3+x^2} = 0 \Rightarrow \lim_{x\to 0}\sqrt{x^3+x^2}\sin\frac{121}{x} = 0$

02 04

(c) $\lim_{x\to 0} \frac{\tan(121x)+\sin x}{x-x^2}$

$= \lim_{x\to 0} \frac{\frac{\sin(121x)}{\cos(121x)} + \sin x}{x(1-x)} = \lim_{x\to 0}\left[\frac{\sin(121x)}{x}\cdot\frac{1}{(1-x)\cos(121x)} + \frac{\sin x}{x}\cdot\frac{1}{1-x}\right]$

$= \lim_{x\to 0}\left[\underbrace{\frac{121\sin(121x)}{121x}}_{121}\cdot\underbrace{\frac{1}{(1-x)\cos(121x)}}_{1} + \underbrace{\frac{\sin x}{x}}_{1}\cdot\underbrace{\frac{1}{1-x}}_{1}\right]$

$= 121\cdot 1 + 1\cdot 1 = 122$

08

(8+8+8) 3. Compute the derivatives of the following functions if they exist (do not simplify the results).

(a) $y = \cos\left(x^2 \cot\sqrt{x}\right)$

$$\frac{dy}{dx} = -\sin\left(x^2\cot\sqrt{x}\right)\cdot\left(2x\cot\sqrt{x} + x^2\cdot\left(-\csc^2\sqrt{x}\cdot\frac{1}{2\sqrt{x}}\right)\right)$$

02 02 02 02

(b) $x^2\cos y + \sin 2y = xy$, $\frac{dy}{dx} = ?$

02 02 02 02

$$2x\cos y + x^2\cdot(-\sin y)\,y' + \cos(2y)\cdot 2y' = y + xy'$$

$$\Rightarrow \left(-x^2\sin y + 2\cos(2y) - x\right)y' = y - 2x\cos y$$

$$\Rightarrow y' = \frac{y - 2x\cos y}{2\cos(2y) - x^2\sin y - x}$$

(c) $y = \sqrt{|x|}$, at the point $x = 0$

$$f'(0) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0}\frac{\sqrt{|h|}}{h}$$ 02

- $\lim_{h\to 0^-}\frac{\sqrt{|h|}}{h} \underset{h<0}{=} \lim_{h\to 0^-}\frac{\sqrt{-h}}{h} = \lim_{h\to 0^-}\frac{-1}{\sqrt{-h}} = -\infty$ 02
- $\lim_{h\to 0^+}\frac{\sqrt{|h|}}{h} \underset{h>0}{=} \lim_{h\to 0^+}\frac{\cancel{\sqrt{h}}}{\cancel{h}} = \lim_{h\to 0^+}\frac{1}{\sqrt{h}} = \infty$ 02

The given func. is not diff. at $x=0$. 02

(16) 4. Use the Intermediate Value Theorem to prove that the equation $e^{-x} = \ln x$ has at least one solution in the interval $(1,2)$.

Let $f(x) = e^{-x} - \ln x$. f is cont on $[1,2]$. 05

$f(1) = \frac{1}{e} - \ln 1 = \frac{1}{e} > 0$

$f(2) = \frac{1}{e^2} - \ln 2 < 0$

f differs in sign 06

By IVT there exist at least one $x_0 \in (1,2)$ such that $f(x_0) = 0$. This means that $e^{-x_0} = \ln x_0$ for some $x_0 \in (1,2)$. 05

(10) 5. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined by $f(x) = \arctan x + 3x - 2$. Find $\lim_{x\to 0}\frac{f^2(x) - f^2(0)}{x}$.

$$\lim_{x\to 0}\frac{f^2(x) - f^2(0)}{x} = \lim_{x\to 0}\left[\frac{f(x) - f(0)}{x - 0}\right]\cdot\lim_{x\to 0}\left[f(x) + f(0)\right] = f'(0)\cdot 2f(0)$$ 05

$f'(x) = \frac{1}{1+x^2} + 3 \Rightarrow f'(0) = 4$; $f(0) = \arctan 0 - 2 = -2$

$$\therefore \lim_{x\to 0}\frac{f^2(x) - f^2(0)}{x} = 4\cdot 2\cdot(-2) = -16$$ 05

(Bonus) 6. Determine whether the statement is true or false. No justification is needed.

(a) A vertical line intersects the graph of a function at most once. T F

(b) If $\lim_{x\to a} f(x) = 3$ and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a}\frac{f(x)}{g(x)}$ does not exist. T F

(c) If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a}\frac{f(x)}{g(x)}$ does not exist. T F

(d) If the line $x = 2$ is a vertical asymptote of $y = f(x)$, then f is not defined at 2. T F

(e) If $f(2) > 0$ and $f(4) < 0$, then there exists a number c between 2 and 4 such that $f(c) = 0$. T F

(f) If $\lim_{x\to 1} f(x) = \infty$ and $\lim_{x\to 1} g(x) = \infty$, then $\lim_{x\to 1}[f(x) - g(x)] = 0$. T F

(g) If f is not continuous at 4, then $f(4)$ is not defined. T F

(h) If f and g are even, then $f + g$ is even. T F

(i) If $\lim_{x\to 0} f(x)\,g(x)$ exists, then the limit must be $f(0)\,g(0)$. T F

(j) If the line $y = 3$ is a horizontal asymptote of $y = f(x)$, then this line does not cross the graph of $y = f(x)$. T F