İstanbul Commerce University
Faculty of Engineering
MAT121-Mathematical Analysis I
Summer School
Midterm Exam

Name-Surname: Solution Key
ID Number:
Department:

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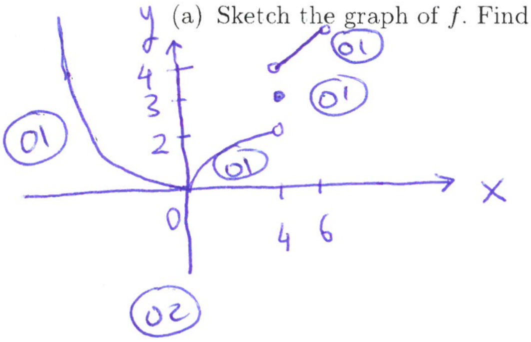
Attention. The exam duration is **90** minutes. The exam is out of **104 points** and consists of **4 questions** which have more than one part. Write your name, surname and student number on top of the first page. Please read the questions carefully and write your answers neatly under the corresponding questions. Show all your work. Correct answers without sufficient explanation might **not** get full credit. Calculators are **not** allowed, nor needed. Good luck.

(10+6+8+6) 1. Let $f(x) = \begin{cases} x^2, & x \leq 0, \\ \sqrt{x}, & 0 < x < 4 \\ x, & 4 < x < 6 \\ 3, & x = 4 \end{cases}$.

(a) Sketch the graph of f. Find the domain and range of f.



$\text{Dom} f = (-\infty, 6)$ 02

$\text{Range} f = [0, \infty)$ 02

(b) Does $\lim_{x \to 4} f(x)$ exist? Explain your answer.

No. Because there is a break at $x = 4$.
02 04

(c) Is f continuous at $x = 0$ and at $x = 4$? Explain your answer.

At $x=0$ f is cont. since there is no hole or break here.
02 02

But at $x=4$, f is not cont. since f has no limit here.
02 02

02 (d) Is f differentiable at $x = 0$? Explain your answer.

No. There is a sharp corner at 0. So f is not diff. at $x=0$.
04

(8+8+8+8) 2. Evaluate the following limits or explain why they do not exist (L'Hopital's rule is not allowed).

(a) $\lim_{x\to 0} \frac{|\sin x|}{x}$

02 $\lim_{x\to 0^-} \frac{-\sin x}{x} = -\lim_{x\to 0^-} \frac{\sin x}{x} = -1$

02 $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$ $\not\equiv$ So limit d.n.e.

04

(b) $\lim_{x\to\infty} \frac{\cos(121x)}{\ln x}$

$-1 \leq \cos(121x) \leq 1 \quad \forall x > 0$

$-\frac{1}{\ln x} \leq \frac{\cos(121x)}{\ln x} \leq \frac{1}{\ln x} \quad \forall x > 0$ 02

$\lim_{x\to\infty} -\frac{1}{\ln x} = \lim_{x\to\infty} \frac{1}{\ln x} = 0$. By Sandwich Thm

02

$\lim_{x\to\infty} \frac{\cos(121x)}{\ln x} = 0$.

04

(c) $\lim_{x\to 0} \frac{x^2 \tan x}{3\sin(x^3)}$

$\lim_{x\to 0} \frac{x^2 \tan x}{3\sin(x^3)} = \lim_{x\to 0} \frac{x^3}{\sin(x^3)} \cdot \frac{\sin x}{x} \cdot \frac{1}{3\cos x}$ 02

$= \lim_{x\to 0} \frac{1}{\frac{\sin(x^3)}{x^3}} \cdot \frac{\sin x}{x} \cdot \frac{1}{3\cos x}$

$= 1 \cdot 1 \cdot \frac{1}{3} = 1/3$. 06

(d) $\lim_{x\to\infty} x^2 - \sqrt{x^4+7x^2+1}$ $\dfrac{(x^2+\sqrt{x^4+7x^2+1})}{}$ $= \lim_{x\to\infty} \dfrac{x^4 - (x^4+7x^2+1)}{x^2+\sqrt{x^4+7x^2+1}}$ 02

$$= \lim_{x\to\infty} \frac{-7x^2-1}{x^2+\sqrt{x^4(1+\frac{7}{x^2}+\frac{1}{x^4})}} = \lim_{x\to\infty} \frac{\cancel{x^2}(-7-1/x^2)}{\cancel{x^2}(1+\sqrt{1+\frac{7}{x^2}+\frac{1}{x^4}})}$$

$$= \frac{-7}{1+\sqrt{1}} = -\frac{7}{2} \,//$$ 06

(8+8+8+8) 3. This question has four unrelated parts.

(a) Find $f'(x)$ if $f(x) = \sqrt{2x^2+2x+1} + \frac{1}{\sqrt[3]{x^4}}$.

$$f(x) = (2x^2+2x+1)^{1/2} + x^{-4/3}$$

$$f'(x) = \frac{1}{2}(2x^2+2x+1)^{-1/2}(4x+2) + \left(-\frac{4}{3}\right)x^{-7/3}$$

02 02 02

02

(b) Let $f(x) = x^7 + 2x + 3$. Find $(f^{-1})'(3)$.

$$f(x) = x^7+2x+3 = 3 \Rightarrow x(x^6+2) = 0 \Rightarrow x = 0$$

$$\therefore f^{-1}(3) = 0$$ 02

$$f'(x) = 7x^6+2 \Rightarrow f'(0) = 2$$ 02

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(0)} = \frac{1}{2} \,//$$ 04

(c) Let $g(x) = 1 + \sqrt{x}$ and $(f \circ g)(x) = 3 + 2\sqrt{x} + x$. Find $f'(2)$.

$$f'(g(x))\, g'(x) = \frac{1}{\sqrt{x}} + 1$$

$$f'(g(x)) \cdot \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}} + 1 \Rightarrow f'(g(x)) = \frac{\frac{1}{\sqrt{x}} + 1}{\frac{1}{2\sqrt{x}}} \quad 02$$

02

$$\left(\begin{array}{l} g(x) = 2 \Rightarrow 1 + \sqrt{x} = 2 \\ \qquad \Rightarrow x = 1 \\ \therefore g(1) = 2 \end{array} \right)$$

$$\therefore f'(g(1)) = \frac{1+1}{\frac{1}{2}} = 4$$

$$\therefore f'(2) = 4 \quad 04$$

(d) If $\sin(xy) = x^2 \cos y$, find $\dfrac{dy}{dx}$ at the point $(2, \pi/2)$.

02

$$\cos(xy)\,(y + x\,y') = 2x\cos y - x^2 \sin y\; y' \quad 02$$

Put $(x, y) = (2, \pi/2)$ into the eqn

$$\underbrace{\cos\pi}_{-1}\left(\frac{\pi}{2} + 2y'\right) = 4\,\cancel{\cos\frac{\pi}{2}}^{\,0} - 4\,\cancel{\sin\frac{\pi}{2}}^{\,1}\, y' \quad 02$$

$$-\frac{\pi}{2} - 2y' = -4y'$$

$$-\frac{\pi}{2} = -2y'$$

$$\frac{\pi}{4} = y' \quad 02$$

(10) 4. Let

$$f(x) = \begin{cases} \frac{\sin^2 x}{x^2 - x}, & x \neq 0, \\ a, & x = 0 \end{cases}$$

Determine the values of a so that f is continuous at $x = 0$.

$$f(0) = a \;; \quad \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin^2 x}{x^2 - x} \quad 02$$

02

$$= \lim_{x \to 0} \frac{\sin^2 x}{x(x-1)}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x^2}\, \frac{x}{x-1} =$$

$$= \lim_{x \to 0} \Big(\underbrace{\frac{\sin x}{x}}_{1}\Big)^2 \frac{x}{x-1} = 1 \cdot 0 = 0$$

So a must be equal to 0. 06