MAT101 – Mathematics-I

Limits

Infinite Limits and Limits at Infinity

In this section, we consider two new types of limits: infinite limits and limits at infinity. Infinite limits and vertical asymptotes are used to describe the behavior of functions that are unbounded near $x = a$. Limits at infinity and horizontal asymptotes are used to describe the behavior of functions as x assumes arbitrarily large positive values or arbitrarily large negative values.

Infinite Limits

The graph of $f(x) = \dfrac{1}{x - 1}$ (Fig. 1) indicates that

$$\lim_{x \to 1^+} \frac{1}{x - 1}$$

does not exist. There does not exist a real number L that the values of $f(x)$ approach as x approaches 1 from the right. Instead, as x approaches 1 from the right, the values of $f(x)$ are positive and become larger and larger; that is, $f(x)$ increases without bound (Table 1). We express this behavior symbolically as

$$\lim_{x \to 1^+} \frac{1}{x - 1} = \infty \quad \text{or} \quad f(x) = \frac{1}{x - 1} \to \infty \quad \text{as} \quad x \to 1^+ \qquad (1)$$

Infinite Limits and Limits at Infinity

Since ∞ is a not a real number, *the limit in (1) does not exist*. We are using the symbol ∞ to describe the manner in which the limit fails to exist, and we call this situation an **infinite limit**. If x approaches 1 from the left, the values of $f(x)$ are negative and become larger and larger in absolute value; that is, $f(x)$ decreases through negative values without bound (Table 2). We express this behavior symbolically as

$$\lim_{x \to 1^-} \frac{1}{x-1} = -\infty \quad \text{or} \quad f(x) = \frac{1}{x-1} \to -\infty \quad \text{as} \quad x \to 1^- \qquad (2)$$

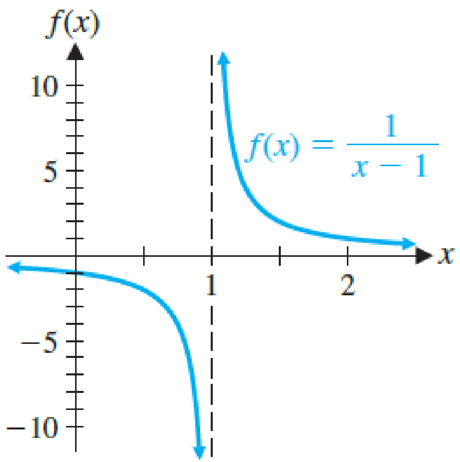


Table 1

x	$f(x) = \frac{1}{x-1}$
1.1	10
1.01	100
1.001	1,000
1.0001	10,000
1.00001	100,000
1.000001	1,000,000

Table 2

x	$f(x) = \frac{1}{x-1}$
0.9	−10
0.99	−100
0.999	−1,000
0.9999	−10,000
0.99999	−100,000
0.999999	−1,000,000

The one-sided limits in (1) and (2) describe the behavior of the graph as $x \to 1$ (Fig. 1). Does the two-sided limit of $f(x)$ as $x \to 1$ exist? No, because neither of the one-sided limits exists. Also, there is no reasonable way to use the symbol ∞ to describe the behavior of $f(x)$ as $x \to 1$ on both sides of 1. We say that

$$\lim_{x \to 1} \frac{1}{x-1} \text{ does not exist}$$

Infinite Limits and Limits at Infinity

DEFINITION Infinite Limits and Vertical Asymptotes

The vertical line $x = a$ is a **vertical asymptote** for the graph of $y = f(x)$ if

$$f(x) \to \infty \quad \text{or} \quad f(x) \to -\infty \quad \text{as} \quad x \to a^+ \quad \text{or} \quad x \to a^-$$

[That is, if $f(x)$ either increases or decreases without bound as x approaches a from the right or from the left].

Locating Vertical Asymptotes

How do we locate vertical asymptotes? If f is a polynomial function, then $\lim_{x \to a} f(x)$ is equal to the real number $f(a)$ [Theorem 3, Section 10.1]. So *a polynomial function has no vertical asymptotes*. Similarly (again by Theorem 3, Section 10.1), *a vertical asymptote of a rational function can occur only at a zero of its denominator*. Theorem 1 provides a simple procedure for locating the vertical asymptotes of a rational function.

THEOREM 1 Locating Vertical Asymptotes of Rational Functions

If $f(x) = n(x)/d(x)$ is a rational function, $d(c) = 0$ and $n(c) \neq 0$, then the line $x = c$ is a vertical asymptote of the graph of f.

Infinite Limits and Limits at Infinity

If $f(x) = n(x)/d(x)$ and both $n(c) = 0$ and $d(c) = 0$, then the limit of $f(x)$ as x approaches c involves an indeterminate form and Theorem 1 does not apply:

$$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{n(x)}{d(x)} \quad \frac{0}{0} \text{ indeterminate form}$$

Algebraic simplification is often useful in this situation.

EXAMPLE 1 Locating Vertical Asymptotes Let $f(x) = \dfrac{x^2 + x - 2}{x^2 - 1}$.

Describe the behavior of f at each zero of the denominator. Use ∞ and $-\infty$ when appropriate. Identify all vertical asymptotes.

SOLUTION Let $n(x) = x^2 + x - 2$ and $d(x) = x^2 - 1$. Factoring the denominator, we see that

$$d(x) = x^2 - 1 = (x - 1)(x + 1)$$

has two zeros: $x = -1$ and $x = 1$.

Infinite Limits and Limits at Infinity

SOLUTION

First, we consider $x = -1$. Since $d(-1) = 0$ and $n(-1) = -2 \neq 0$, Theorem 1 tells us that the line $x = -1$ is a vertical asymptote. So at least one of the one-sided limits at $x = -1$ must be either ∞ or $-\infty$. Examining tables of values of f for x near -1 or a graph on a graphing calculator will show which is the case. From Tables 3 and 4, we see that

$$\lim_{x \to -1^-} \frac{x^2 + x - 2}{x^2 - 1} = -\infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{x^2 + x - 2}{x^2 - 1} = \infty$$

Table 3

x	$f(x) = \dfrac{x^2 + x - 2}{x^2 - 1}$
-1.1	-9
-1.01	-99
-1.001	-999
-1.0001	$-9,999$
-1.00001	$-99,999$

Table 4

x	$f(x) = \dfrac{x^2 + x - 2}{x^2 - 1}$
-0.9	11
-0.99	101
-0.999	$1,001$
-0.9999	$10,001$
-0.99999	$100,001$

Infinite Limits and Limits at Infinity

SOLUTION

Now we consider the other zero of $d(x)$, $x = 1$. This time $n(1) = 0$ and Theorem 1 does not apply. We use algebraic simplification to investigate the behavior of the function at $x = 1$:

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - 1} \qquad \frac{0}{0} \text{ indeterminate form}$$

$$= \lim_{x \to 1} \frac{(x - 1)(x + 2)}{(x - 1)(x + 1)}$$

$$= \lim_{x \to 1} \frac{x + 2}{x + 1}$$

$$= \frac{3}{2}$$

Since the limit exists as x approaches 1, f does not have a vertical asymptote at $x = 1$. The graph of f (Fig. 2) shows the behavior at the vertical asymptote $x = -1$ and also at $x = 1$.

Matched Problem 1 Let $f(x) = \dfrac{x - 3}{x^2 - 4x + 3}$.

Describe the behavior of f at each zero of the denominator. Use ∞ and $-\infty$ when appropriate. Identify all vertical asymptotes.

Infinite Limits and Limits at Infinity

EXAMPLE 2 Locating Vertical Asymptotes Let $f(x) = \dfrac{x^2 + 20}{5(x-2)^2}$.

Describe the behavior of f at each zero of the denominator. Use ∞ and $-\infty$ when appropriate. Identify all vertical asymptotes.

SOLUTION Let $n(x) = x^2 + 20$ and $d(x) = 5(x-2)^2$. The only zero of $d(x)$ is $x = 2$. Since $n(2) = 24 \neq 0$, f has a vertical asymptote at $x = 2$ (Theorem 1). Tables 5 and 6 show that $f(x) \to \infty$ as $x \to 2$ from either side, and we have

$$\lim_{x\to 2^+} \frac{x^2+20}{5(x-2)^2} = \infty \quad \text{and} \quad \lim_{x\to 2^-} \frac{x^2+20}{5(x-2)^2} = \infty$$

Table 5

x	$f(x) = \dfrac{x^2 + 20}{5(x-2)^2}$
2.1	488.2
2.01	48,080.02
2.001	4,800,800.2

Table 6

x	$f(x) = \dfrac{x^2 + 20}{5(x-2)^2}$
1.9	472.2
1.99	47,920.02
1.999	4,799,200.2

Infinite Limits and Limits at Infinity

SOLUTION

The denominator d has no other zeros, so f does not have any other vertical asymptotes. The graph of f (Fig. 3) shows the behavior at the vertical asymptote $x = 2$. Because the left- and right-hand limits are both infinite, we write

$$\lim_{x \to 2} \frac{x^2 + 20}{5(x - 2)^2} = \infty$$

Matched Problem 2 Let $f(x) = \dfrac{x - 1}{(x + 3)^2}$.

Describe the behavior of f at each zero of the denominator. Use ∞ and $-\infty$ when appropriate. Identify all vertical asymptotes.

Infinite Limits and Limits at Infinity

CONCEPTUAL INSIGHT

When is it correct to say that a limit does not exist, and when is it correct to use $\pm\infty$? It depends on the situation. Table 7 lists the infinite limits that we discussed in Examples 1 and 2.

Table 7

Right-Hand Limit	Left-Hand Limit	Two-Sided Limit
$\lim_{x\to-1^+}\frac{x^2+x-2}{x^2-1}=\infty$	$\lim_{x\to-1^-}\frac{x^2+x-2}{x^2-1}=-\infty$	$\lim_{x\to-1}\frac{x^2+x-2}{x^2-1}$ does not exist
$\lim_{x\to-2^+}\frac{x^2+20}{5(x-2)^2}=\infty$	$\lim_{x\to2^-}\frac{x^2+20}{5(x-2)^2}=\infty$	$\lim_{x\to2}\frac{x^2+20}{5(x-2)^2}=\infty$

The instructions in Examples 1 and 2 said that we should use infinite limits to describe the behavior at vertical asymptotes. If we had been asked to *evaluate* the limits, with no mention of ∞ or asymptotes, then the correct answer would be that **all of these limits do not exist**. Remember, ∞ is a symbol used to describe the behavior of functions at vertical asymptotes.

Infinite Limits and Limits at Infinity

Limits at Infinity

The symbol ∞ can also be used to indicate that an independent variable is increasing or decreasing without bound. We write $x \longrightarrow \infty$ to indicate that x is increasing without bound through positive values and $x \longrightarrow -\infty$ to indicate that x is decreasing without bound through negative values. We begin by considering power functions of the form x^p and $1/x^p$ where p is a positive real number.

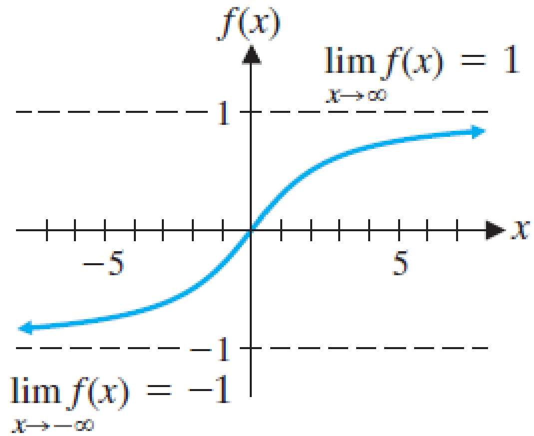
If p is a positive real number, then x^p increases as x increases. There is no upper bound on the values of x^p. We indicate this behavior by writing

$$\lim_{x \to \infty} x^p = \infty \quad \text{or} \quad x^p \longrightarrow \infty \quad \text{as} \quad x \longrightarrow \infty$$

Since the reciprocals of very large numbers are very small numbers, it follows that $1/x^p$ approaches 0 as x increases without bound. We indicate this behavior by writing

$$\lim_{x \to \infty} \frac{1}{x^p} = 0 \quad \text{or} \quad \frac{1}{x^p} \longrightarrow 0 \quad \text{as} \quad x \longrightarrow \infty$$

Infinite Limits and Limits at Infinity



For the function g in Figure 4, the line $y = 0$ (the x axis) is called a *horizontal asymptote*. In general, a line $y = b$ is a **horizontal asymptote** of the graph of $y = f(x)$ if $f(x)$ approaches b as either x increases without bound or x decreases without bound. Symbolically, $y = b$ is a horizontal asymptote if either

$$\lim_{x \to -\infty} f(x) = b \quad \text{or} \quad \lim_{x \to \infty} f(x) = b$$

In the first case, the graph of f will be close to the horizontal line $y = b$ for large (in absolute value) negative x. In the second case, the graph will be close to the horizontal line $y = b$ for large positive x. Figure 5 shows the graph of a function with two horizontal asymptotes: $y = 1$ and $y = -1$.

Infinite Limits and Limits at Infinity

THEOREM 2 Limits of Power Functions at Infinity

If p is a positive real number and k is any real number except 0, then

1. $\lim_{x \to -\infty} \frac{k}{x^p} = 0$ **2.** $\lim_{x \to \infty} \frac{k}{x^p} = 0$

3. $\lim_{x \to -\infty} kx^p = \pm\infty$ **4.** $\lim_{x \to \infty} kx^p = \pm\infty$

provided that x^p is a real number for negative values of x. The limits in 3 and 4 will be either $-\infty$ or ∞, depending on k and p.

EXAMPLE 3 Limit of a Polynomial Function at Infinity Let $p(x) = 2x^3 - x^2 - 7x + 3$. Find the limit of $p(x)$ as x approaches ∞ and as x approaches $-\infty$.

SOLUTION Since limits of power functions of the form $1/x^p$ approach 0 as x approaches ∞ or $-\infty$, it is convenient to work with these reciprocal forms whenever possible. If we factor out the term involving the highest power of x, then we can write $p(x)$ as

$$p(x) = 2x^3\left(1 - \frac{1}{2x} - \frac{7}{2x^2} + \frac{3}{2x^3}\right)$$

Infinite Limits and Limits at Infinity

$$\lim_{x\to\infty}\left(1-\frac{1}{2x}-\frac{7}{2x^2}+\frac{3}{2x^3}\right) = 1-0-0+0 \quad = 1$$

For large values of x,

$$\left(1-\frac{1}{2x}-\frac{7}{2x^2}+\frac{3}{2x^3}\right) \approx 1$$

and

$$p(x) = 2x^3\left(1-\frac{1}{2x}-\frac{7}{2x^2}+\frac{3}{2x^3}\right) \approx 2x^3$$

Since $2x^3 \to \infty$ as $x \to \infty$, it follows that

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} 2x^3 = \infty$$

Similarly, $2x^3 \to -\infty$ as $x \to -\infty$ implies that

$$\lim_{x\to-\infty} p(x) = \lim_{x\to-\infty} 2x^3 = -\infty$$

So the behavior of $p(x)$ for large values is the same as the behavior of the highest-degree term, $2x^3$.

Infinite Limits and Limits at Infinity

The term with highest degree in a polynomial is called the **leading term**. In the solution to Example 3, the limits at infinity of $p(x) = 2x^3 - x^2 - 7x + 3$ were the same as the limits of the leading term $2x^3$. Theorem 3 states that this is true for any polynomial of degree greater than or equal to 1.

THEOREM 3 Limits of Polynomial Functions at Infinity

If

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,\ a_n \neq 0,\ n \geq 1$$

then

$$\lim_{x \to \infty} p(x) = \lim_{x \to \infty} a_n x^n = \pm\infty$$

and

$$\lim_{x \to -\infty} p(x) = \lim_{x \to -\infty} a_n x^n = \pm\infty$$

Each limit will be either $-\infty$ or ∞, depending on a_n and n.

Infinite Limits and Limits at Infinity

EXAMPLE 4 End Behavior of a Polynomial Give a pair of limit expressions that describe the end behavior of each polynomial.

(A) $p(x) = 3x^3 - 500x^2$ (B) $p(x) = 3x^3 - 500x^4$

SOLUTION

(A) By Theorem 3,

$$\lim_{x \to \infty} (3x^3 - 500x^2) = \lim_{x \to \infty} 3x^3 = \infty$$ Right end behavior

and

$$\lim_{x \to -\infty} (3x^3 - 500x^2) = \lim_{x \to -\infty} 3x^3 = -\infty$$ Left end behavior

(B) By Theorem 3,

$$\lim_{x \to \infty} (3x^3 - 500x^4) = \lim_{x \to \infty} (-500x^4) = -\infty$$ Right end behavior

and

$$\lim_{x \to -\infty} (3x^3 - 500x^4) = \lim_{x \to -\infty} (-500x^4) = -\infty$$ Left end behavior

Infinite Limits and Limits at Infinity

Finding Horizontal Asymptotes

THEOREM 4 Limits of Rational Functions at Infinity and Horizontal Asymptotes of Rational Functions

(A) If $f(x) = \dfrac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0}$, $a_m \neq 0, b_n \neq 0$

then $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \dfrac{a_m x^m}{b_n x^n}$ and $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \dfrac{a_m x^m}{b_n x^n}$

(B) There are three possible cases for these limits:

1. If $m < n$, then $\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 0$, and the line $y = 0$ (the x axis) is a horizontal asymptote of $f(x)$.
2. If $m = n$, then $\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = \dfrac{a_m}{b_n}$, and the line $y = \dfrac{a_m}{b_n}$ is a horizontal asymptote of $f(x)$.
3. If $m > n$, then each limit will be ∞ or $-\infty$, depending on m, n, a_m, and b_n, and $f(x)$ does not have a horizontal asymptote.

Infinite Limits and Limits at Infinity

Notice that in cases 1 and 2 of Theorem 4, the limit is the same if x approaches ∞ or $-\infty$. So, **a rational function can have at most one horizontal asymptote** (see Fig. 6).

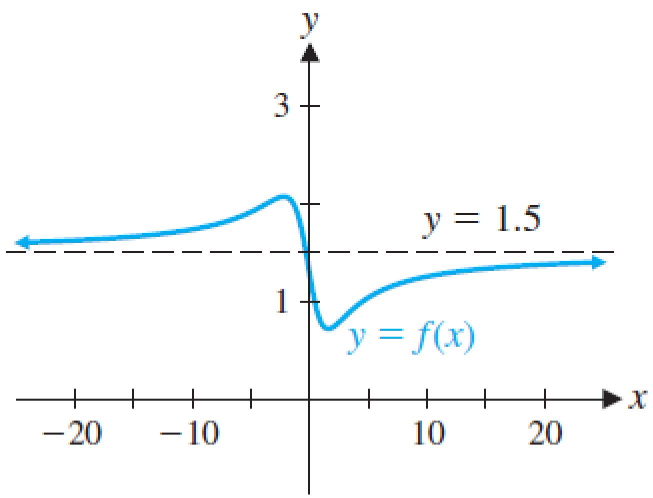


Figure 6 $f(x) = \dfrac{3x^2 - 5x + 9}{2x^2 + 7}$

Infinite Limits and Limits at Infinity

EXAMPLE 5 Finding Horizontal Asymptotes Find all horizontal asymptotes, if any, of each function.

(A) $f(x) = \dfrac{5x^3 - 2x^2 + 1}{4x^3 + 2x - 7}$

(B) $f(x) = \dfrac{3x^4 - x^2 + 1}{8x^6 - 10}$

(C) $f(x) = \dfrac{2x^5 - x^3 - 1}{6x^3 + 2x^2 - 7}$

SOLUTION We will make use of part A of Theorem 4.

(A) $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \dfrac{5x^3 - 2x^2 + 1}{4x^3 + 2x - 7} = \lim_{x \to \infty} \dfrac{5x^3}{4x^3} = \dfrac{5}{4}$

The line $y = 5/4$ is a horizontal asymptote of $f(x)$.

(B) $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \dfrac{3x^4 - x^2 + 1}{8x^6 - 10} = \lim_{x \to \infty} \dfrac{3x^4}{8x^6} = \lim_{x \to \infty} \dfrac{3}{8x^2} = 0$

The line $y = 0$ (the x axis) is a horizontal asymptote of $f(x)$.

(C) $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \dfrac{2x^5 - x^3 - 1}{6x^3 + 2x^2 - 7} = \lim_{x \to \infty} \dfrac{2x^5}{6x^3} = \lim_{x \to \infty} \dfrac{x^2}{3} = \infty$

The function $f(x)$ has no horizontal asymptotes.

Infinite Limits and Limits at Infinity

Matched Problem 5 Find all horizontal asymptotes, if any, of each function.

(A) $f(x) = \dfrac{4x^3 - 5x + 8}{2x^4 - 7}$ (B) $f(x) = \dfrac{5x^6 + 3x}{2x^5 - x - 5}$

(C) $f(x) = \dfrac{2x^3 - x + 7}{4x^3 + 3x^2 - 100}$

EXAMPLE 6 Find all vertical and horizontal asymptotes of the function

$$f(x) = \frac{2x^2 - 5}{x^2 + 5x + 4}$$

SOLUTION Let $n(x) = 2x^2 - 5$ and $d(x) = x^2 + 5x + 4 = (x + 1)(x + 4)$. The denominator $d(x) = 0$ at $x = -1$ and $x = -4$. Since the numerator $n(x)$ is not zero at these values of x [$n(-1) = -3$ and $n(-4) = 27$], by Theorem 1 there are two vertical asymptotes of f: the line $x = -1$ and the line $x = -4$. Since

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^2 - 5}{x^2 + 5x + 4} = \lim_{x\to\infty} \frac{2x^2}{x^2} = 2$$

the horizontal asymptote is the line $y = 2$ (Theorem 3).

Infinite Limits and Limits at Infinity

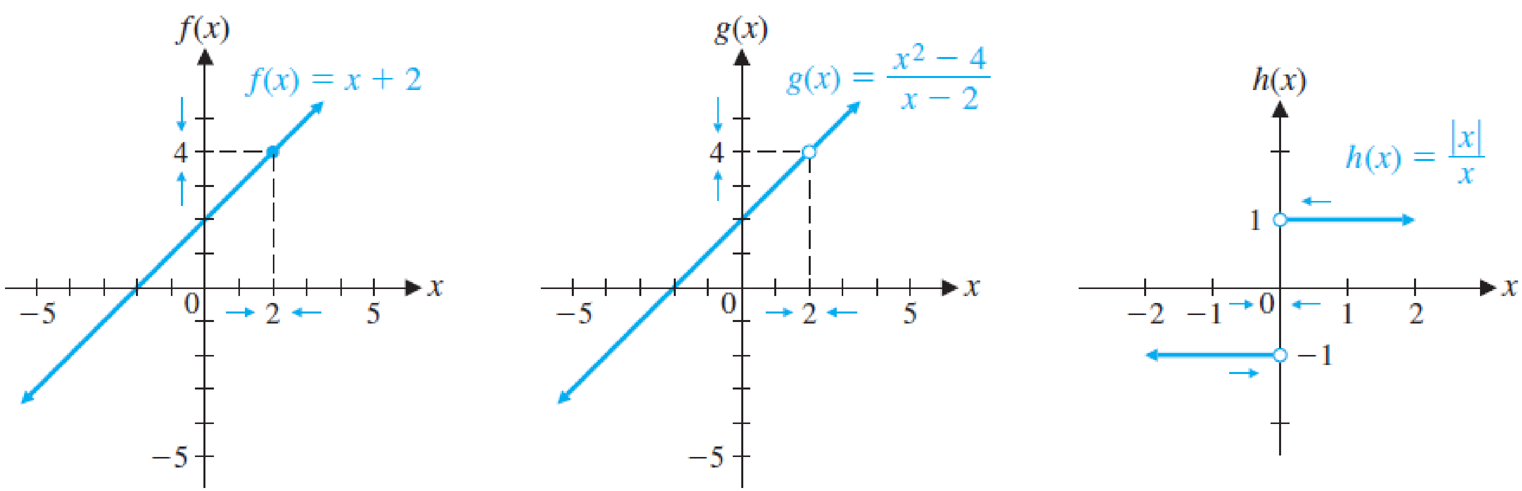
Matched Problem 6 Find all vertical and horizontal asymptotes of the function

$$f(x) = \frac{x^2 - 9}{x^2 - 4}.$$

Continuity

Continuity

Compare the graphs shown in Figure 1. Notice that two of the graphs are broken; that is, they cannot be drawn without lifting a pen off the paper. Informally, a function is *continuous over an interval* if its graph over the interval can be drawn without removing a pen from the paper. A function whose graph is broken (disconnected) at $x = c$ is said to be *discontinuous* at $x = c$. Function f (Fig. 1A) is continuous for all x. Function g (Fig. 1B) is discontinuous at $x = 2$ but is continuous over any interval that does not include 2. Function h (Fig. 1C) is discontinuous at $x = 0$ but is continuous over any interval that does not include 0.



(A) $\lim_{x\to 2} f(x) = 4$
$f(2) = 4$

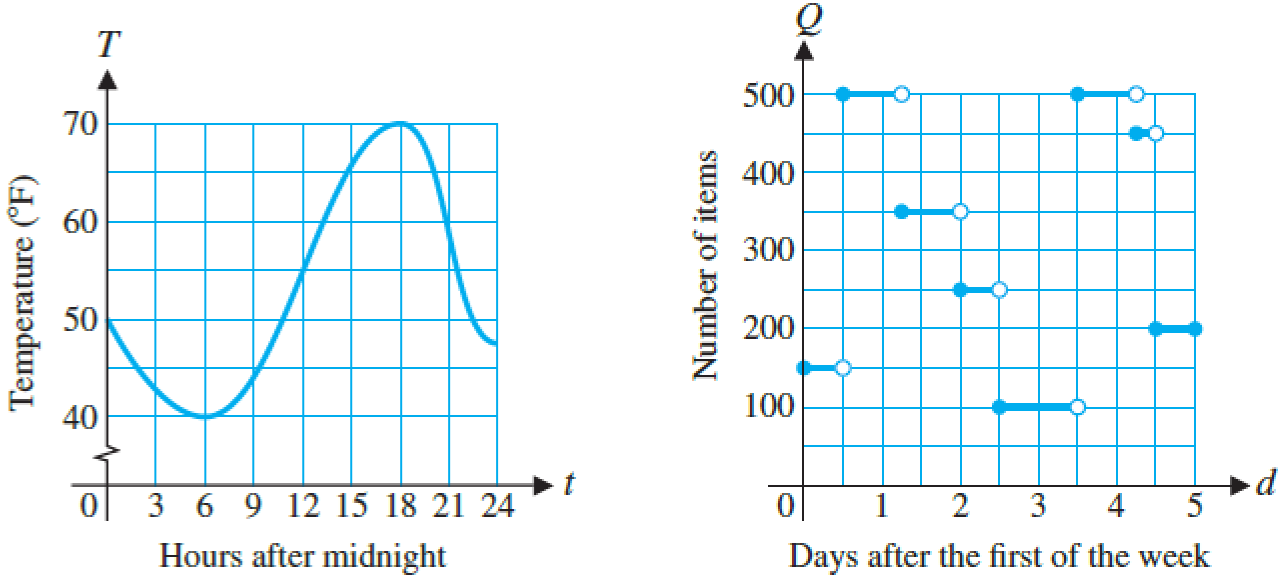
(B) $\lim_{x\to 2} g(x) = 4$
$g(2)$ is not defined

(C) $\lim_{x\to 0} h(x)$ does not exist
$h(0)$ is not defined

Figure 1

Continuity

Most graphs of natural phenomena are continuous, whereas many graphs in business and economics applications have discontinuities. Figure 2A illustrates temperature variation over a 24-hour period—a continuous phenomenon. Figure 2B illustrates warehouse inventory over a 1-week period—a discontinuous phenomenon.



(A) Temperature for a 24-hour period

(B) Inventory in a warehouse during 1 week

Figure 2

Continuity

The preceding discussion leads to the following formal definition of continuity:

DEFINITION Continuity

A function f is **continuous at the point $x = c$ if**

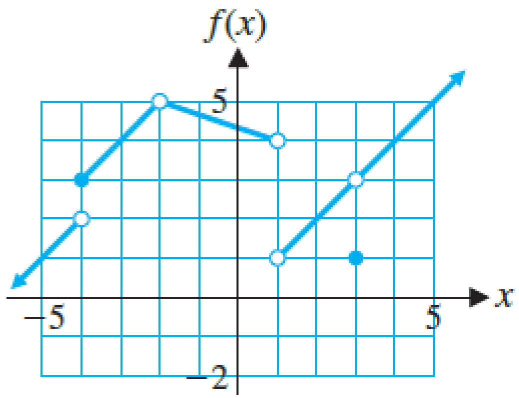
1. $\lim_{x \to c} f(x)$ exists **2.** $f(c)$ exists **3.** $\lim_{x \to c} f(x) = f(c)$

A function is **continuous on the open interval*** **(a, b)** if it is continuous at each point on the interval.

If one or more of the three conditions in the definition fails, then the function is **discontinuous** at $x = c$.

Continuity

EXAMPLE 1 Continuity of a Function Defined by a Graph Use the definition of continuity to discuss the continuity of the function whose graph is shown in Figure 3.



SOLUTION We begin by identifying the points of discontinuity. Examining the graph, we see breaks or holes at $x = -4, -2, 1$, and 3. Now we must determine which conditions in the definition of continuity are not satisfied at each of these points. In each case, we find the value of the function and the limit of the function at the point in question.

Discontinuity at $x = -4$:

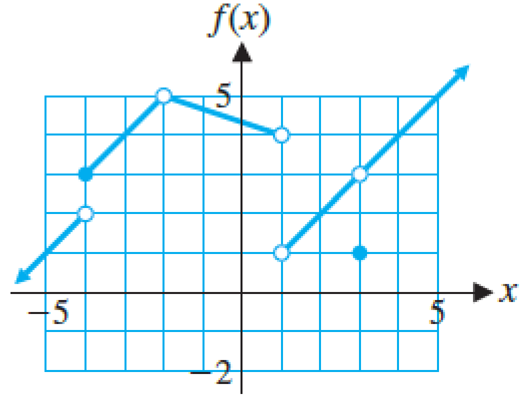
$$\lim_{x \to -4^-} f(x) = 2$$
$$\lim_{x \to -4^+} f(x) = 3$$

Since the one-sided limits are different, the limit does not exist (Section 10.1).

$$\lim_{x \to -4} f(x) \text{ does not exist}$$
$$f(-4) = 3$$

So, f is not continuous at $x = -4$ because condition 1 is not satisfied.

Continuity



Discontinuity at $x = -2$:

$$\lim_{x \to -2^-} f(x) = 5$$

$$\lim_{x \to -2^+} f(x) = 5$$

$$\lim_{x \to -2} f(x) = 5$$

$$f(-2) \text{ does not exist}$$

The hole at $(-2, 5)$ indicates that 5 is not the value of f at -2. Since there is no solid dot elsewhere on the vertical line $x = -2$, $f(-2)$ is not defined.

So even though the limit as x approaches -2 exists, f is not continuous at $x = -2$ because condition 2 is not satisfied.

Discontinuity at $x = 1$:

$$\lim_{x \to 1^-} f(x) = 4$$

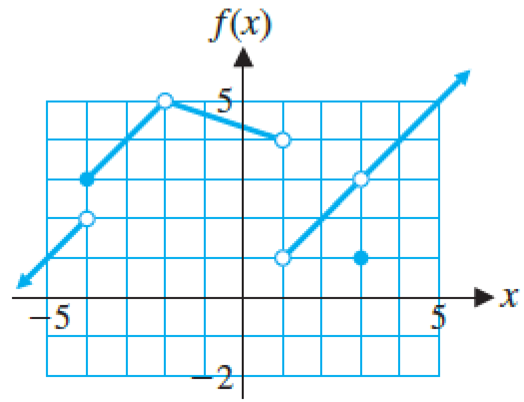
$$\lim_{x \to 1^+} f(x) = 1$$

$$\lim_{x \to 1} f(x) \text{ does not exist}$$

$$f(1) \text{ does not exist}$$

This time, f is not continuous at $x = 1$ because neither of conditions 1 and 2 is satisfied.

Continuity



Discontinuity at $x = 3$:

$$\lim_{x \to 3^-} f(x) = 3$$ The solid dot at (3, 1) indicates that $f(3) = 1$.

$$\lim_{x \to 3^+} f(x) = 3$$

$$\lim_{x \to 3} f(x) = 3$$

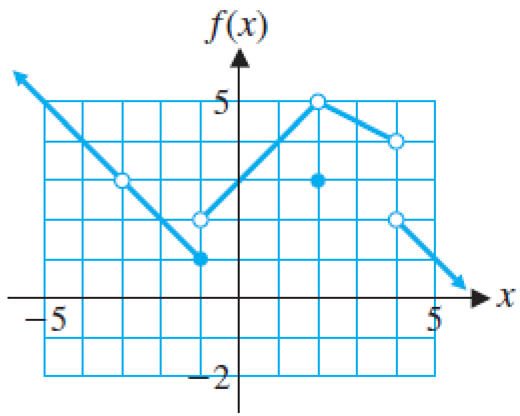
$$f(3) = 1$$

Conditions 1 and 2 are satisfied, but f is not continuous at $x = 3$ because condition 3 is not satisfied.

Having identified and discussed all points of discontinuity, we can now conclude that f is continuous except at $x = -4, -2, 1$, and 3.

Continuity

Matched Problem 1 Use the definition of continuity to discuss the continuity of the function whose graph is shown in Figure 4.



EXAMPLE 2 Continuity of Functions Defined by Equations Using the definition of continuity, discuss the continuity of each function at the indicated point(s).

(A) $f(x) = x + 2$ at $x = 2$

(B) $g(x) = \dfrac{x^2 - 4}{x - 2}$ at $x = 2$

(C) $h(x) = \dfrac{|x|}{x}$ at $x = 0$ and at $x = 1$

Continuity

SOLUTION

(A) f is continuous at $x = 2$, since

$$\lim_{x \to 2} f(x) = 4 = f(2)$$ See Figure 1A.

(B) g is not continuous at $x = 2$, since $g(2) = 0/0$ is not defined (see Fig. 1B).

(C) h is not continuous at $x = 0$, since $h(0) = |0|/0$ is not defined; also, $\lim_{x \to 0} h(x)$ does not exist.

h is continuous at $x = 1$, since

$$\lim_{x \to 1} \frac{|x|}{x} = 1 = h(1)$$ See Figure 1C.

Matched Problem 2 Using the definition of continuity, discuss the continuity of each function at the indicated point(s).

(A) $f(x) = x + 1$ at $x = 1$

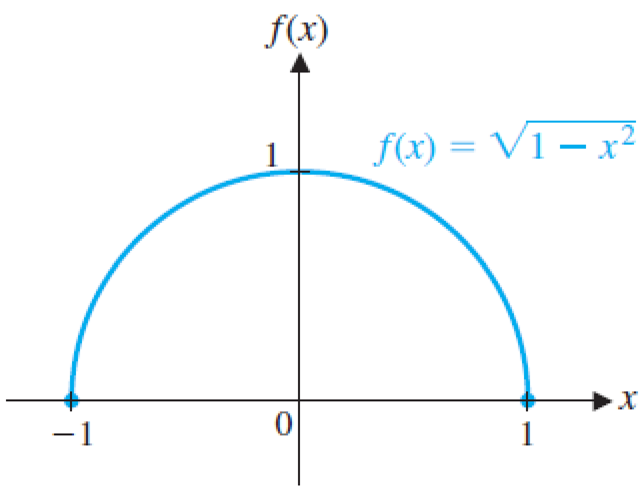
(B) $g(x) = \dfrac{x^2 - 1}{x - 1}$ at $x = 1$

(C) $h(x) = \dfrac{x - 2}{|x - 2|}$ at $x = 2$ and at $x = 0$

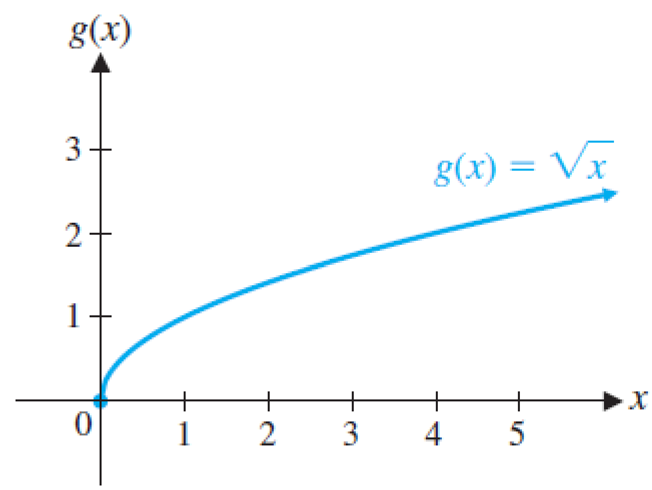
Continuity

We can also talk about one-sided continuity, just as we talked about one-sided limits. For example, a function is said to be **continuous on the right** at $x = c$ if $\lim_{x \to c^+} f(x) = f(c)$ and **continuous on the left** at $x = c$ if $\lim_{x \to c^-} f(x) = f(c)$. A function is **continuous on the closed interval [a, b]** if it is continuous on the open interval (a, b) and is continuous both on the right at a and on the left at b.

Figure 5A illustrates a function that is continuous on the closed interval $[-1, 1]$. Figure 5B illustrates a function that is continuous on the half-closed interval $[0, \infty)$.



(A) f is continuous on the closed interval $[-1, 1]$



(B) g is continuous on the half-closed interval $[0, \infty)$

Figure 5 **Continuity on closed and half-closed intervals**

Continuity

THEOREM 1 Continuity Properties of Some Specific Functions

(A) A constant function $f(x) = k$, where k is a constant, is continuous for all x.
$f(x) = 7$ is continuous for all x.

(B) For n a positive integer, $f(x) = x^n$ is continuous for all x.
$f(x) = x^5$ is continuous for all x.

(C) A polynomial function is continuous for all x.
$2x^3 - 3x^2 + x - 5$ is continuous for all x.

(D) A rational function is continuous for all x except those values that make a denominator 0.
$\frac{x^2 + 1}{x - 1}$ is continuous for all x except $x = 1$, a value that makes the denominator 0.

(E) For n an odd positive integer greater than 1, $\sqrt[n]{f(x)}$ is continuous wherever $f(x)$ is continuous.
$\sqrt[3]{x^2}$ is continuous for all x.

(F) For n an even positive integer, $\sqrt[n]{f(x)}$ is continuous wherever $f(x)$ is continuous and nonnegative.
$\sqrt[4]{x}$ is continuous on the interval $[0, \infty)$.

Continuity

EXAMPLE 3 Using Continuity Properties Using Theorem 1 and the general properties of continuity, determine where each function is continuous.

(A) $f(x) = x^2 - 2x + 1$

(B) $f(x) = \dfrac{x}{(x + 2)(x - 3)}$

(C) $f(x) = \sqrt[3]{x^2 - 4}$

(D) $f(x) = \sqrt{x - 2}$

SOLUTION

(A) Since f is a polynomial function, f is continuous for all x.

(B) Since f is a rational function, f is continuous for all x except -2 and 3 (values that make the denominator 0).

(C) The polynomial function $x^2 - 4$ is continuous for all x. Since $n = 3$ is odd, f is continuous for all x.

(D) The polynomial function $x - 2$ is continuous for all x and nonnegative for $x \geq 2$. Since $n = 2$ is even, f is continuous for $x \geq 2$, or on the interval $[2, \infty)$.

Continuity

Matched Problem 3 Using Theorem 1 and the general properties of continuity, determine where each function is continuous.

(A) $f(x) = x^4 + 2x^2 + 1$

(B) $f(x) = \dfrac{x^2}{(x + 1)(x - 4)}$

(C) $f(x) = \sqrt{x - 4}$

(D) $f(x) = \sqrt[3]{x^3 + 1}$

Continuity

Problem

Natural-gas rates. Table 1 shows the rates for natural gas charged by the Middle Tennessee Natural Gas Utility District during summer months. The base charge is a fixed monthly charge, independent of the amount of gas used per month.

Table 1 **Summer (May–September)**

Base charge	\$5.00
First 50 therms	0.63 per therm
Over 50 therms	0.45 per therm

(A) Write a piecewise definition of the monthly charge $S(x)$ for a customer who uses x therms* in a summer month.

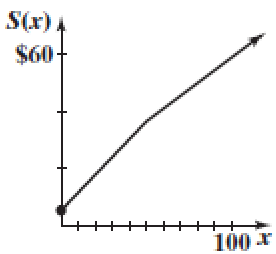
(B) Graph $S(x)$.

(C) Is $S(x)$ continuous at $x = 50$? Explain.

SOLUTION

(A) $S(x) = \begin{cases} 5 + 0.63x & \text{if } 0 \le x \le 50 \\ 14 + 0.45x & \text{if } 50 < x \end{cases}$

(B)



(C) Yes

Continuity

Problem **Income.** A personal-computer salesperson receives a base salary of \$1,000 per month and a commission of 5% of all sales over \$10,000 during the month. If the monthly sales are \$20,000 or more, then the salesperson is given an additional \$500 bonus. Let $E(s)$ represent the person's earnings per month as a function of the monthly sales s.

(A) Graph $E(s)$ for $0 \le s \le 30{,}000$.

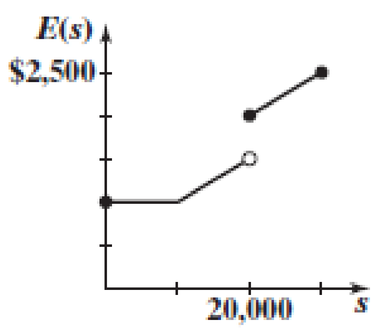
(B) Find $\lim_{s \to 10{,}000} E(s)$ and $E(10{,}000)$.

(C) Find $\lim_{s \to 20{,}000} E(s)$ and $E(20{,}000)$.

(D) Is E continuous at $s = 10{,}000$? At $s = 20{,}000$?

SOLUTION

(A)



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(B) $\lim_{x \to 10{,}000} E(s) = \$1{,}000$; $E(10{,}000) = \$1{,}000$

(C) $\lim_{x \to 20{,}000} E(s)$ does not exist; $E(20{,}000) = \$2{,}000$

(D) Yes; no

Problem

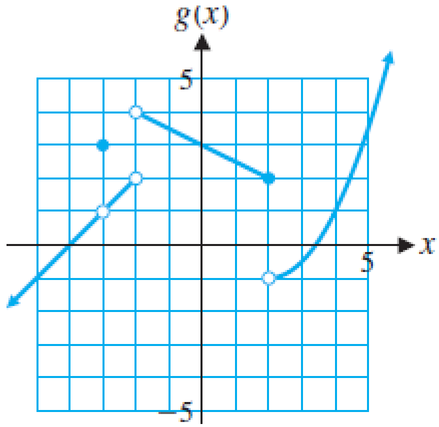

Figure for 23–30

23. $g(-3.1)$

24. $g(-2.1)$

25. $g(1.9)$

26. $g(-1.9)$

27. (A) $\lim_{x\to -3^-} g(x)$ (B) $\lim_{x\to -3^+} g(x)$

(C) $\lim_{x\to -3} g(x)$ (D) $g(-3)$

(E) Is g continuous at $x = -3$? Explain.

28. (A) $\lim_{x\to -2^-} g(x)$ (B) $\lim_{x\to -2^+} g(x)$

(C) $\lim_{x\to -2} g(x)$ (D) $g(-2)$

(E) Is g continuous at $x = -2$? Explain.

29. (A) $\lim_{x\to 2^-} g(x)$ (B) $\lim_{x\to 2^+} g(x)$

(C) $\lim_{x\to 2} g(x)$ (D) $g(2)$

(E) Is g continuous at $x = 2$? Explain.

30. (A) $\lim_{x\to 4^-} g(x)$ (B) $\lim_{x\to 4^+} g(x)$

(C) $\lim_{x\to 4} g(x)$ (D) $g(4)$

(E) Is g continuous at $x = 4$? Explain.