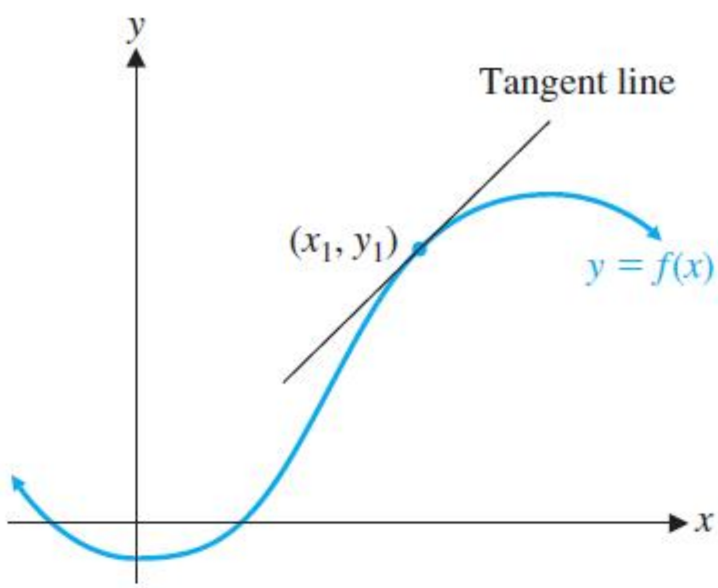
MAT101 – Mathematics-I

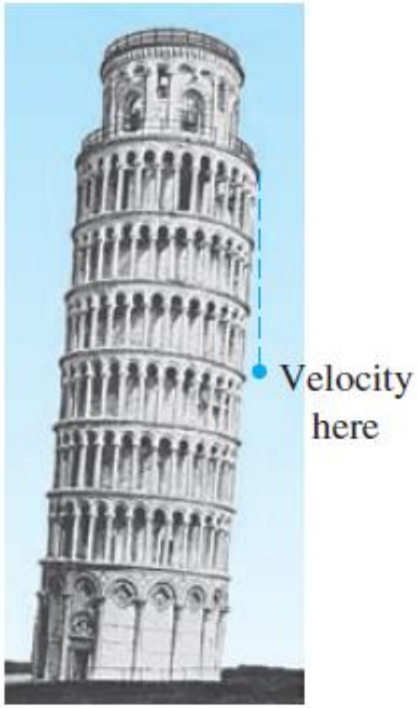
Derivative

Two basic problems

We will now make use of the limit concepts developed in Sections 10.1, 10.2, and 10.3 to solve the two important problems illustrated in Figure 1. The solution of each of these apparently unrelated problems involves a common concept called the *derivative*.



(A) Find the equation of the tangent line at (x_1, y_1) given $y = f(x)$



(B) Find the instantaneous velocity of a falling object

Rate of Change

We will define the concepts of average rate of change and instantaneous rate of change more generally, and will apply them in situations that are unrelated to velocity.

DEFINITION Average Rate of Change

For $y = f(x)$, the **average rate of change from $x = a$ to $x = a + h$** is

$$\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h} \qquad h \neq 0 \qquad (1)$$

Note that the numerator and denominator in (1) are differences, so (1) is a **difference quotient**

Rate of Change: Revenue Analysis

EXAMPLE Revenue Analysis The revenue (in dollars) from the sale of x plastic planter boxes is given by

$$R(x) = 20x - 0.02x^2 \qquad 0 \le x \le 1{,}000$$

and is graphed in Figure 2.

(A) What is the change in revenue if production is changed from 100 planters to 400 planters?

(B) What is the average rate of change in revenue for this change in production?

SOLUTION

(A) The change in revenue is given by

$$\begin{aligned} R(400) - R(100) &= 20(400) - 0.02(400)^2 - [20(100) - 0.02(100)^2] \\ &= 4{,}800 - 1{,}800 = \$3{,}000 \end{aligned}$$

Increasing production from 100 planters to 400 planters will increase revenue by \$3,000.

Rate of Change: Revenue Analysis

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and is graphed in Figure 2.

(A) What is the change in revenue if production is changed from 100 planters to 400 planters?

(B) What is the average rate of change in revenue for this change in production?

SOLUTION

(B) To find the average rate of change in revenue, we divide the change in revenue by the change in production:

$$\frac{R(400) - R(100)}{400 - 100} = \frac{3{,}000}{300} = \$10$$

The average rate of change in revenue is \$10 per planter when production is increased from 100 to 400 planters.

Rate of Change

DEFINITION Instantaneous Rate of Change

For $y = f(x)$, the **instantaneous rate of change at $x = a$** is

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{2}$$

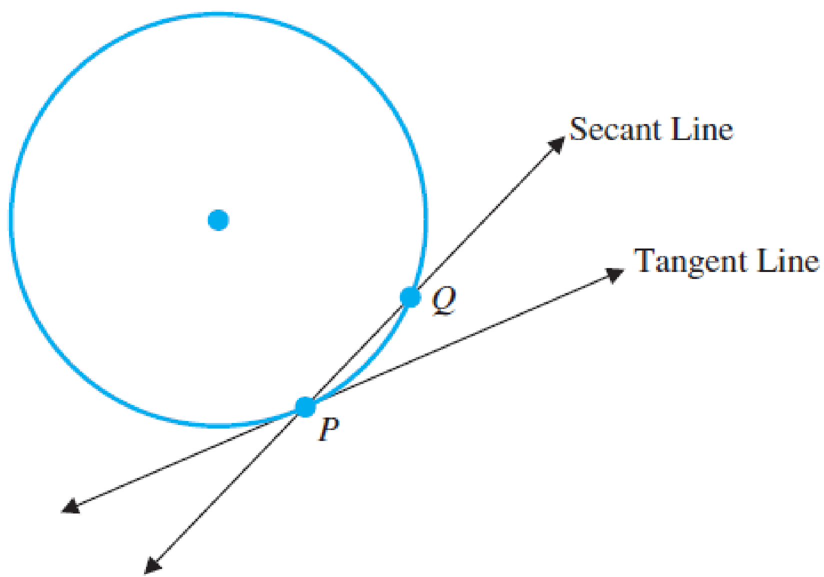
if the limit exists.

The adjective *instantaneous* is often omitted with the understanding that the phrase **rate of change** always refers to the instantaneous rate of change and not the average rate of change. Similarly, **velocity** always refers to the instantaneous rate of change of distance with respect to time.

Tangent and Secant Line

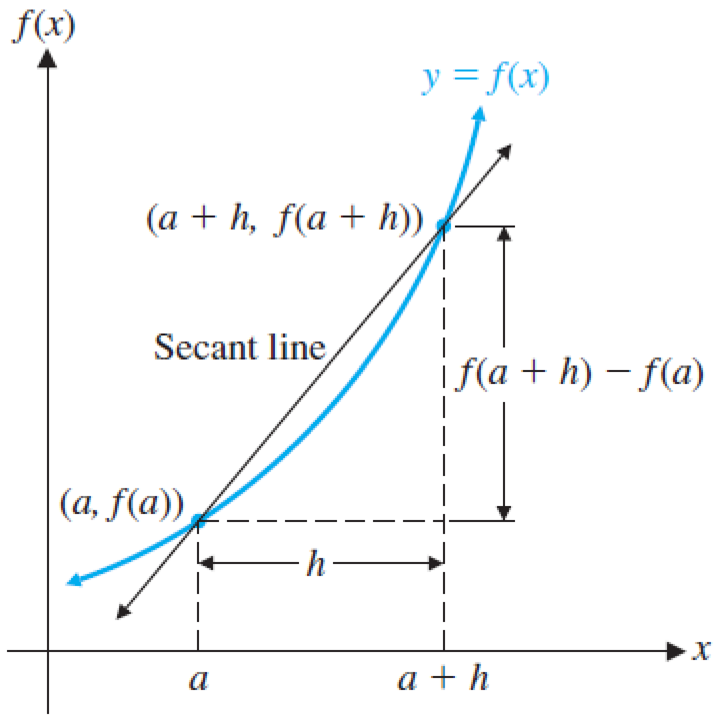
So far, our interpretations of the difference quotient have been numerical in nature. Now we want to consider a geometric interpretation.

In geometry, a line that intersects a circle in two points is called a *secant line*, and a line that intersects a circle in exactly one point is called a *tangent line* (Fig. 4, page 550). If the point Q in Figure 4 is moved closer and closer to the point P, then the angle between the secant line PQ and the tangent line at P gets smaller and smaller. We will generalize the geometric concepts of secant line and tangent line of a circle and will use them to study graphs of functions.



Secant line and tangent line of a circle

Slope of the Secant Line



A line through two points on the graph of a function is called a **secant line**. If $(a, f(a))$ and $(a + h, f(a + h))$ are two points on the graph of $y = f(x)$, then we can use the slope formula from Section 1.2 to find the slope of the secant line through these points (Fig. 5).

$$\textbf{Slope of secant line} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a + h) - f(a)}{(a + h) - a}$$

$$= \frac{f(a + h) - f(a)}{h} \quad \text{Difference quotient}$$

Slope of the Secant Line: Example

EXAMPLE Slope of a Secant Line Given $f(x) = x^2$,

(A) Find the slope of the secant line for $a = 1$ and $h = 2$ and 1, respectively. Graph $y = f(x)$ and the two secant lines.

(B) Find and simplify the slope of the secant line for $a = 1$ and h any nonzero number.

(C) Find the limit of the expression in part (B).

(D) Discuss possible interpretations of the limit in part (C).

SOLUTION

(A) For $a = 1$ and $h = 2$, the secant line goes through $(1, f(1)) = (1, 1)$ and $(3, f(3)) = (3, 9)$, and its slope is

$$\frac{f(1 + 2) - f(1)}{2} = \frac{3^2 - 1^2}{2} = 4$$

For $a = 1$ and $h = 1$, the secant line goes through $(1, f(1)) = (1, 1)$ and $(2, f(2)) = (2, 4)$, and its slope is

$$\frac{f(1 + 1) - f(1)}{1} = \frac{2^2 - 1^2}{1} = 3$$

Slope of the Secant Line: Example

SOLUTION

(B) For $a = 1$ and h any nonzero number, the secant line goes through $(1, f(1)) = (1, 1)$ and $(1 + h, f(1 + h)) = (1 + h, (1 + h)^2)$, and its slope is

$$\frac{f(1+h) - f(1)}{h} = \frac{(1+h)^2 - 1^2}{h}$$

Square the binomial.

$$= \frac{1 + 2h + h^2 - 1}{h}$$

Combine like terms and factor the numerator.

$$= \frac{h(2+h)}{h}$$

Cancel.

$$= 2 + h \qquad h \neq 0$$

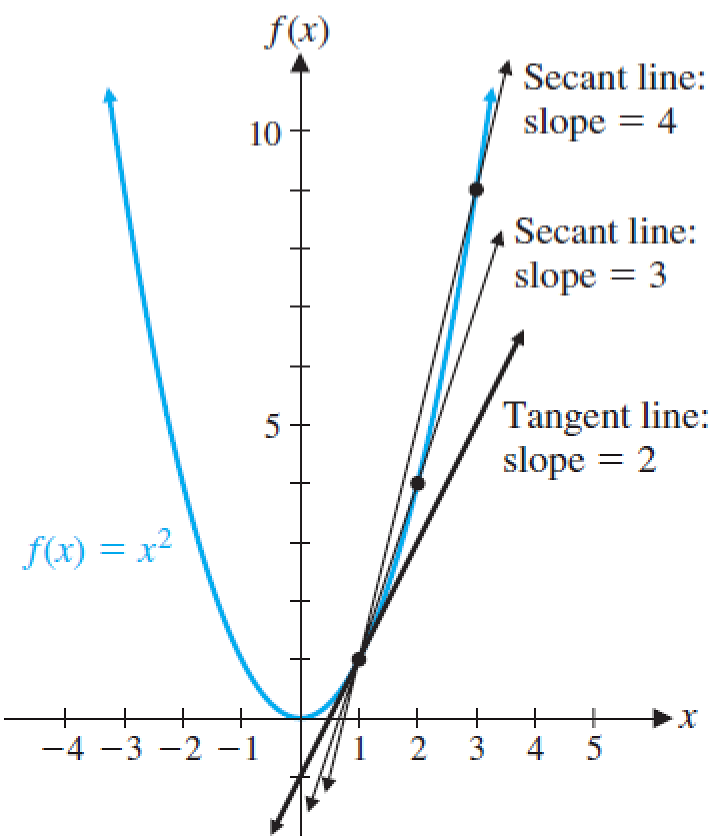
(C) The limit of the secant line slope from part (B) is

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} (2 + h)$$

$$= 2$$

Slope of the Secant Line: Example

SOLUTION



(D) In part (C), we saw that the limit of the slopes of the secant lines through the point (1, *f*(1)) is 2.

If we graph the line through (1, *f*(1)) with slope 2, then this line is the limit of the secant lines.

The slope obtained from the limit of slopes of secant lines is called the *slope of the graph* at $x = 1$.

The line through the point (1, *f*(1)) with this slope is called the *tangent line.*

Slope of the Tangent Line and Derivative

The ideas introduced in the preceding example are summarized next:

DEFINITION Slope of a Graph and Tangent Line

Given $y = f(x)$, the **slope of the graph** at the point $(a, f(a))$ is given by

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{3}$$

provided the limit exists. In this case, the **tangent line** to the graph is the line through $(a, f(a))$ with slope given by (3).

DEFINITION The Derivative

For $y = f(x)$, we define the **derivative of f at x**, denoted $f'(x)$, by

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \quad \text{if the limit exists}$$

If $f'(x)$ exists for each x in the open interval (a, b), then f is said to be **differentiable** over (a, b).

Derivative: Summary

SUMMARY Interpretations of the Derivative

The derivative of a function f is a new function f'. The domain of f' is a subset of the domain of f. The derivative has various applications and interpretations, including the following:

1. *Slope of the tangent line.* For each x in the domain of f', $f'(x)$ is the slope of the line tangent to the graph of f at the point $(x, f(x))$.
2. *Instantaneous rate of change.* For each x in the domain of f', $f'(x)$ is the instantaneous rate of change of $y = f(x)$ with respect to x.
3. *Velocity.* If $f(x)$ is the position of a moving object at time x, then $v = f'(x)$ is the velocity of the object at that time.

Derivative: Example

EXAMPLE Finding a Derivative Find $f'(x)$, the derivative of f at x, for $f(x) = \sqrt{x} + 2$.

SOLUTION We use the four-step process to find $f'(x)$.

Step 1 Find $f(x + h)$.

$$f(x + h) = \sqrt{x + h} + 2$$

Step 2 Find $f(x + h) - f(x)$.

$$\begin{aligned} f(x + h) - f(x) &= \sqrt{x + h} + 2 - (\sqrt{x} + 2) \quad \text{Combine like terms.} \\ &= \sqrt{x + h} - \sqrt{x} \end{aligned}$$

Derivative: Example

SOLUTION

Step 3 Find $\dfrac{f(x+h)-f(x)}{h}$.

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
&= \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \frac{1}{\sqrt{x+h}+\sqrt{x}} \qquad h \neq 0
\end{aligned}$$

We rationalize the numerator (Appendix A, Section A.6) to change the form of this fraction.

Combine like terms.

Cancel.

Derivative: Example

SOLUTION

Step 4 Find $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$.

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{\sqrt{x + h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \qquad x > 0$$

So the derivative of $f(x) = \sqrt{x} + 2$ is $f'(x) = 1/(2\sqrt{x})$, a new function. The domain of f is $[0, \infty)$. Since $f'(0)$ is not defined, the domain of f' is $(0, \infty)$, a subset of the domain of f.

Matched Problem Find $f'(x)$ for $f(x) = \sqrt{x + 4}$.

Derivative Example: Sales Analysis

EXAMPLE Sales Analysis A company's total sales (in millions of dollars) t months from now are given by $S(t) = \sqrt{t} + 2$. Find and interpret $S(25)$ and $S'(25)$. Use these results to estimate the total sales after 26 months and after 27 months.

SOLUTION The total sales function S has the same form as the function f in Example 7. Only the letters used to represent the function and the independent variable have been changed. It follows that S' and f' also have the same form:

$$S(t) = \sqrt{t} + 2 \qquad f(x) = \sqrt{x} + 2$$

$$S'(t) = \frac{1}{2\sqrt{t}} \qquad f'(x) = \frac{1}{2\sqrt{x}}$$

Evaluating S and S' at $t = 25$, we have

$$S(25) = \sqrt{25} + 2 = 7 \qquad S'(25) = \frac{1}{2\sqrt{25}} = 0.1$$

So 25 months from now, the total sales will be \$7 million and will be increasing at the rate of \$0.1 million (\$100,000) per month. If this instantaneous rate of change of sales remained constant, the sales would grow to \$7.1 million after 26 months, \$7.2 million after 27 months, and so on. Even though $S'(t)$ is not a constant function in this case, these values provide useful estimates of the total sales.

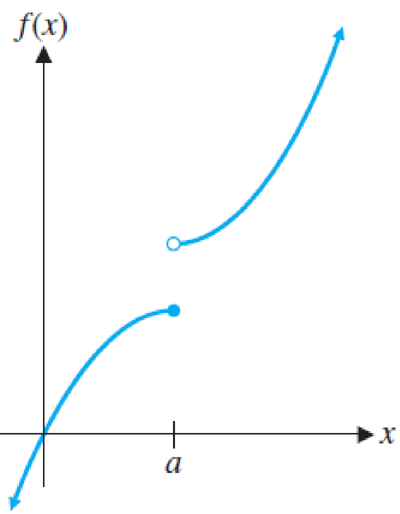
Nonexistence of the Derivative

Nonexistence of the Derivative

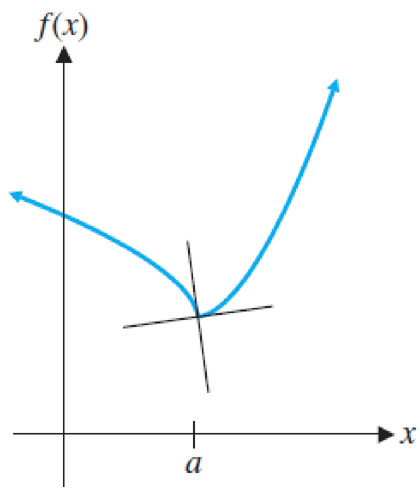
The existence of a derivative at $x = a$ depends on the existence of a limit at $x = a$, that is, on the existence of

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{4}$$

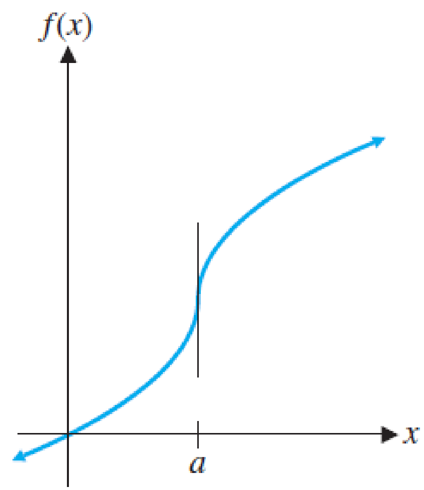
If the limit does not exist at $x = a$, we say that the function f is **nondifferentiable at** $\boldsymbol{x = a}$, or $\boldsymbol{f'(\mathrm{a})}$ **does not exist.**



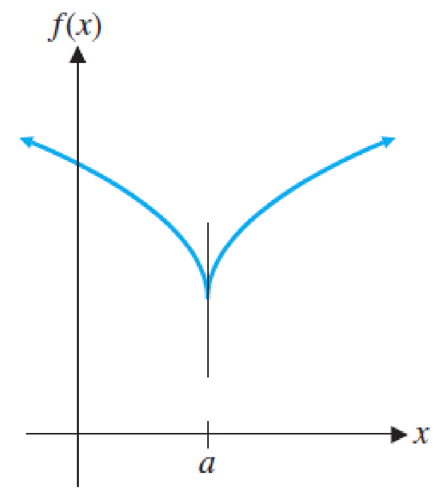
(A) Not continuous at $x = a$



(B) Graph has sharp corner at $x = a$



(C) Vertical tangent at $x = a$



(D) Vertical tangent at $x = a$

Nonexistence of the Derivative

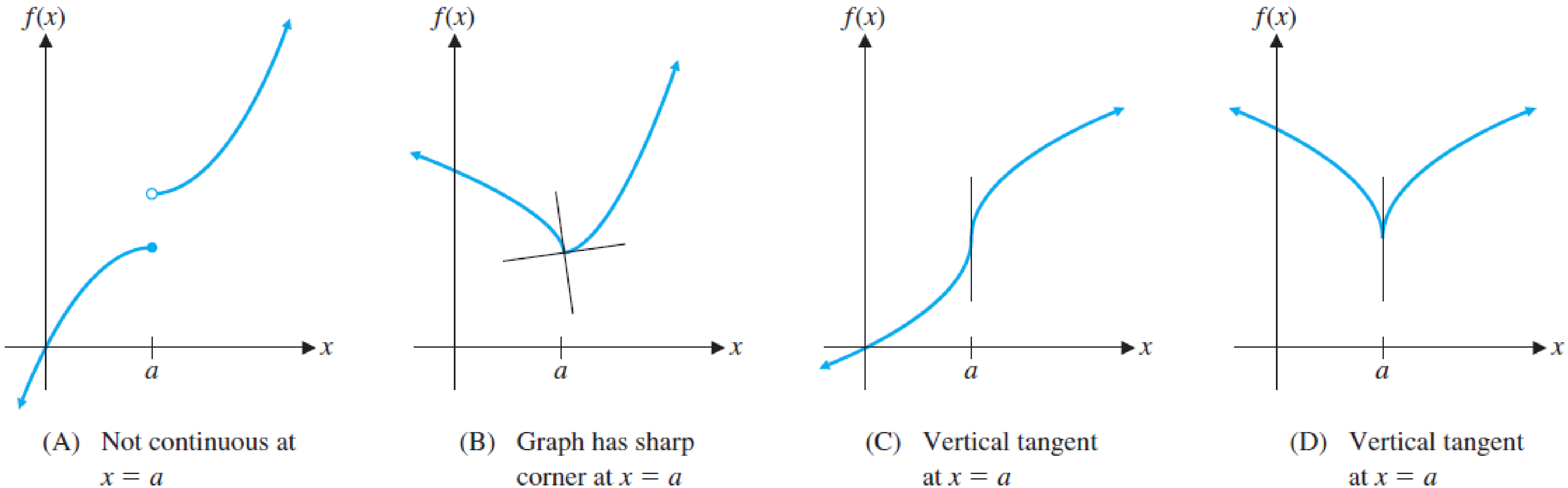


Figure 8 **The function f is nondifferentiable at $x = a$.**

1. If the graph of f has a hole or a break at $x = a$, then $f'(a)$ does not exist (Fig. 8A).
2. If the graph of f has a sharp corner at $x = a$, then $f'(a)$ does not exist, and the graph has no tangent line at $x = a$ (Fig. 8B). (In Fig. 8B, the left- and right-hand derivatives exist but are not equal.)
3. If the graph of f has a vertical tangent line at $x = a$, then $f'(a)$ does not exist (Fig. 8C and D).

Nonexistence of the Derivative

Explore and Discuss

Let $f(x) = |x - 1|$.

(A) Graph f.

(B) Complete the following table:

h	-0.1	-0.01	-0.001	$\rightarrow 0 \leftarrow$	0.001	0.01	0.1
$\dfrac{f(1+h) - f(1)}{h}$	?	?	?	$\rightarrow ? \leftarrow$	?	?	?

(C) Find the following limit if it exists:

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

(D) Use the results of parts (A)–(C) to discuss the existence of $f'(1)$.

(E) Repeat parts (A)–(D) for $\sqrt[3]{x - 1}$.

Basic Differentiation Properties

NOTATION The Derivative

If $y = f(x)$, then

$$f'(x) \qquad y' \qquad \frac{dy}{dx}$$

all represent the derivative of f at x.

THEOREM Constant Function Rule

If $y = f(x) = C$, then

$$f'(x) = 0$$

Also, $y' = 0$ and $dy/dx = 0$.

Note: When we write $C' = 0$ or $\frac{d}{dx}C = 0$, we mean that $y' = \frac{dy}{dx} = 0$ when $y = C$.

Basic Differentiation Properties

THEOREM Power Rule

If $y = f(x) = x^n$, where n is a real number, then

$$f'(x) = nx^{n-1}$$

Also, $y' = nx^{n-1}$ and $dy/dx = nx^{n-1}$.

THEOREM Constant Multiple Property

If $y = f(x) = ku(x)$, then

$$f'(x) = ku'(x)$$

Also,

$$y' = ku' \qquad \frac{dy}{dx} = k\frac{du}{dx}$$

THEOREM Sum and Difference Property

If $y = f(x) = u(x) \pm v(x)$, then

$$f'(x) = u'(x) \pm v'(x)$$

Also,

$$y' = u' \pm v' \qquad \frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$$

Basic Differentiation Properties: Examples

EXAMPLE

(A) If $f(x) = 3x^2 + 2x$, then

$$f'(x) = (3x^2)' + (2x)' = 3(2x) + 2(1) = 6x + 2$$

(B) If $y = 4 + 2x^3 - 3x^{-1}$, then

$$y' = (4)' + (2x^3)' - (3x^{-1})' = 0 + 2(3x^2) - 3(-1)x^{-2} = 6x^2 + 3x^{-2}$$

(C) If $y = \sqrt[3]{w} - 3w$, then

$$\frac{dy}{dw} = \frac{d}{dw}w^{1/3} - \frac{d}{dw}3w = \frac{1}{3}w^{-2/3} - 3 = \frac{1}{3w^{2/3}} - 3$$

(D) $$\frac{d}{dx}\left(\frac{5}{3x^2} - \frac{2}{x^4} + \frac{x^3}{9}\right) = \frac{d}{dx}\frac{5}{3}x^{-2} - \frac{d}{dx}2x^{-4} + \frac{d}{dx}\frac{1}{9}x^3$$

$$= \frac{5}{3}(-2)x^{-3} - 2(-4)x^{-5} + \frac{1}{9}\cdot 3x^2$$

$$= -\frac{10}{3x^3} + \frac{8}{x^5} + \frac{1}{3}x^2$$

Basic Differentiation Properties: Examples

Matched Problem 5 Find

(A) $f'(x)$ for $f(x) = 3x^4 - 2x^3 + x^2 - 5x + 7$

(B) y' for $y = 3 - 7x^{-2}$

(C) $\frac{dy}{dv}$ for $y = 5v^3 - \sqrt[4]{v}$

(D) $\frac{d}{dx}\left(-\frac{3}{4x} + \frac{4}{x^3} - \frac{x^4}{8}\right)$

45. $\frac{d}{dx}\left(\frac{3x^2}{2} - \frac{7}{5x^2}\right)$

46. $\frac{d}{dx}\left(\frac{5x^3}{4} - \frac{2}{5x^3}\right)$

47. $G'(w)$ if $G(w) = \frac{5}{9w^4} + 5\sqrt[3]{w}$

48. $H'(w)$ if $H(w) = \frac{5}{w^6} - 2\sqrt{w}$

49. $\frac{d}{du}(3u^{2/3} - 5u^{1/3})$

Basic Differentiation Properties: Examples

EXAMPLE Rewrite before Differentiating Find the derivative of

$$f(x) = \frac{1 + x^2}{x^4}.$$

SOLUTION It is helpful to rewrite $f(x) = \frac{1 + x^2}{x^4}$, expressing $f(x)$ as the sum of terms, each of which can be differentiated by applying the power rule.

$$f(x) = \frac{1 + x^2}{x^4} \qquad \text{Write as a sum of two terms}$$

$$= \frac{1}{x^4} + \frac{x^2}{x^4} \qquad \text{Write each term as a power of } x$$

$$= x^{-4} + x^{-2}$$

Note that we have rewritten $f(x)$, but we have not used any rules of differentiation. Now, however, we can apply those rules to find the derivative:

$$f'(x) = -4x^{-5} - 2x^{-3}$$

Matched Problem Find the derivative of $f(x) = \frac{5 - 3x + 4x^2}{x}$.

Basic Differentiation Properties: Examples

EXAMPLE Tangents Let $f(x) = x^4 - 6x^2 + 10$.

(A) Find $f'(x)$.

(B) Find the equation of the tangent line at $x = 1$.

(C) Find the values of x where the tangent line is horizontal.

SOLUTION

(A) $f'(x) = (x^4)' - (6x^2)' + (10)'$

$= 4x^3 - 12x$

(B) $y - y_1 = m(x - x_1)$ — $y_1 = f(x_1) = f(1) = (1)^4 - 6(1)^2 + 10 = 5$

$y - 5 = -8(x - 1)$ — $m = f'(x_1) = f'(1) = 4(1)^3 - 12(1) = -8$

$y = -8x + 13$ — Tangent line at $x = 1$

(C) Since a horizontal line has 0 slope, we must solve $f'(x) = 0$ for x:

$f'(x) = 4x^3 - 12x = 0$ — Factor $4x$ out of each term.

$4x(x^2 - 3) = 0$ — Factor the difference of two squares.

$4x(x + \sqrt{3})(x - \sqrt{3}) = 0$ — Use the zero property.

$x = 0, -\sqrt{3}, \sqrt{3}$

Marginal Analysis in Business and Economics

Marginal Analysis in Business and Economics

Marginal Cost, Revenue, and Profit

One important application of calculus to business and economics involves *marginal analysis*. In economics, the word *marginal* refers to a rate of change—that is, to a derivative. Thus, if $C(x)$ is the total cost of producing x items, then $C'(x)$ is called the *marginal cost* and represents the instantaneous rate of change of total cost with respect to the number of items produced. Similarly, the *marginal revenue* is the derivative of the total revenue function, and the *marginal profit* is the derivative of the total profit function.

Marginal Analysis in Business and Economics

DEFINITION Marginal Cost, Revenue, and Profit

If x is the number of units of a product produced in some time interval, then

$$\begin{aligned}
\text{total cost} &= C(x) \\
\textbf{marginal cost} &= C'(x) \\
\text{total revenue} &= R(x) \\
\textbf{marginal revenue} &= R'(x) \\
\text{total profit} &= P(x) = R(x) - C(x) \\
\textbf{marginal profit} &= P'(x) = R'(x) - C'(x) \\
&= (\text{marginal revenue}) - (\text{marginal cost})
\end{aligned}$$

Marginal cost (or revenue or profit) is the instantaneous rate of change of cost (or revenue or profit) relative to production at a given production level.

EXAMPLE Cost Analysis A company manufactures fuel tanks for cars. The total weekly cost (in dollars) of producing x tanks is given by

$$C(x) = 10{,}000 + 90x - 0.05x^2$$

(A) Find the marginal cost function.

(B) Find the marginal cost at a production level of 500 tanks per week.

(C) Interpret the results of part (B).

(D) Find the exact cost of producing the 501st item.

SOLUTION

(A) $C'(x) = 90 - 0.1x$

(B) $C'(500) = 90 - 0.1(500) = \$40$ Marginal cost

(C) At a production level of 500 tanks per week, the total production costs are increasing at the rate of \$40 per tank.

(D)

$$\begin{aligned} C(501) &= 10{,}000 + 90(501) - 0.05(501)^2 \\ &= \$42{,}539.95 && \text{Total cost of producing 501 tanks per week} \\ C(500) &= 10{,}000 + 90(500) - 0.05(500)^2 \\ &= \$42{,}500.00 && \text{Total cost of producing 500 tanks per week} \\ C(501) - C(500) &= 42{,}539.95 - 42{,}500.00 \\ &= \$39.95 && \text{Exact cost of producing the 501st tank} \end{aligned}$$

Marginal Analysis in Business and Economics

Matched Problem A company manufactures automatic transmissions for cars. The total weekly cost (in dollars) of producing x transmissions is given by

$$C(x) = 50{,}000 + 600x - 0.75x^2$$

(A) Find the marginal cost function.

(B) Find the marginal cost at a production level of 200 transmissions per week.

(C) Interpret the results of part (B).

(D) Find the exact cost of producing the 201st transmission.

Marginal Analysis in Business and Economics

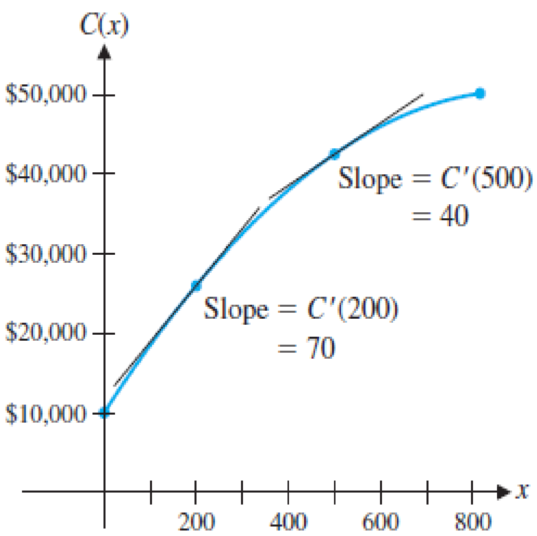
THEOREM Marginal Cost and Exact Cost

If $C(x)$ is the total cost of producing x items, then the marginal cost function approximates the exact cost of producing the $(x + 1)$st item:

Marginal cost Exact cost

$$C'(x) \approx C(x + 1) - C(x)$$

Similar statements can be made for total revenue functions and total profit functions.



CONCEPTUAL INSIGHT

Theorem 1 states that the marginal cost at a given production level x approximates the cost of producing the $(x + 1)$st, or *next*, item. In practice, the marginal cost is used more frequently than the exact cost. One reason for this is that the marginal cost is easily visualized when one is examining the graph of the total cost function. Figure 2 shows the graph of the cost function discussed in Example 1, with tangent lines added at $x = 200$ and $x = 500$. The graph clearly shows that as production increases, the slope of the tangent line decreases. Thus, the cost of producing the next tank also decreases, a desirable characteristic of a total cost function. We will have much more to say about graphical analysis in Chapter 12.

Marginal Analysis in Business and Economics: Example

EXAMPLE Exact Cost and Marginal Cost The total cost of producing x bicycles is given by the cost function

$$C(x) = 10{,}000 + 150x - 0.2x^2$$

(A) Find the exact cost of producing the 121st bicycle.

(B) Use marginal cost to approximate the cost of producing the 121st bicycle.

SOLUTION

(A) The cost of producing 121 bicycles is

$$C(121) = 10{,}000 + 150(121) - 0.2(121)^2 = \$25{,}221.80$$

and the cost of producing 120 bicycles is

$$C(120) = 10{,}000 + 150(120) - 0.2(120)^2 = \$25{,}120.00$$

So the exact cost of producing the 121st bicycle is

$$C(121) - C(120) = \$25{,}221.80 - 25{,}120.00 = \$101.80$$

Marginal Analysis in Business and Economics: Example

EXAMPLE Exact Cost and Marginal Cost The total cost of producing x bicycles is given by the cost function

$$C(x) = 10{,}000 + 150x - 0.2x^2$$

(A) Find the exact cost of producing the 121st bicycle.

(B) Use marginal cost to approximate the cost of producing the 121st bicycle.

SOLUTION

(B) By Theorem 1, the marginal cost function $C'(x)$, evaluated at $x = 120$, approximates the cost of producing the 121st bicycle:

$$C'(x) = 150 - 0.4x$$
$$C'(120) = 150 - 0.4(120) = \$102.00$$

Note that the marginal cost, \$102.00, at a production level of 120 bicycles, is a good approximation to the exact cost, \$101.80, of producing the 121st bicycle.

EXAMPLE Production Strategy A company's market research department recommends the manufacture and marketing of a new headphone set for MP3 players. After suitable test marketing, the research department presents the following **price–demand equation**:

$$x = 10{,}000 - 1{,}000p \quad \text{x is demand at price p.} \tag{1}$$

In the price–demand equation (1), the demand x is given as a function of price p. By solving (1) for p (add $1{,}000p$ to both sides of the equation, subtract x from both sides, and divide both sides by 1,000), we obtain equation (2), in which the price p is given as a function of demand x:

$$p = 10 - 0.001x \tag{2}$$

where x is the number of headphones that retailers are likely to buy at $\$p$ per set.

The financial department provides the **cost function**

$$C(x) = 7{,}000 + 2x \tag{3}$$

where \$7,000 is the estimate of fixed costs (tooling and overhead) and \$2 is the estimate of variable costs per headphone set (materials, labor, marketing, transportation, storage, etc.).

(A) Find the domain of the function defined by the price–demand equation (2).

(B) Find and interpret the marginal cost function $C'(x)$.

(C) Find the revenue function as a function of x and find its domain.

(D) Find the marginal revenue at $x = 2{,}000$, $5{,}000$, and $7{,}000$. Interpret these results.

(E) Graph the cost function and the revenue function in the same coordinate system. Find the intersection points of these two graphs and interpret the results.

(F) Find the profit function and its domain and sketch the graph of the function.

(G) Find the marginal profit at $x = 1{,}000$, $4{,}000$, and $6{,}000$. Interpret these results.

SOLUTION

(A) Since price p and demand x must be nonnegative, we have $x \geq 0$ and

$$p = 10 - 0.001x \geq 0$$
$$10 \geq 0.001x$$
$$10{,}000 \geq x$$

Thus, the permissible values of x are $0 \leq x \leq 10{,}000$.

(B) The marginal cost is $C'(x) = 2$. Since this is a constant, it costs an additional \$2 to produce one more headphone set at any production level.

SOLUTION

(C) The **revenue** is the amount of money R received by the company for manufacturing and selling x headphone sets at \$$p$ per set and is given by

$$R = (\text{number of headphone sets sold})(\text{price per headphone set}) = xp$$

In general, the revenue R can be expressed as a function of p using equation (1) or as a function of x using equation (2). As we mentioned earlier, when using marginal functions, we will always use the number of items x as the independent variable. Thus, the **revenue function** is

$$\begin{aligned} R(x) &= xp = x(10 - 0.001x) \quad \text{Using equation (2)} \qquad (4) \\ &= 10x - 0.001x^2 \end{aligned}$$

Since equation (2) is defined only for $0 \le x \le 10{,}000$, it follows that the domain of the revenue function is $0 \le x \le 10{,}000$.

SOLUTION

(D) The **marginal revenue** is

$$R'(x) = 10 - 0.002x$$

For production levels of $x = 2{,}000$, 5,000, and 7,000, we have

$$R'(2{,}000) = 6 \qquad R'(5{,}000) = 0 \qquad R'(7{,}000) = -4$$

This means that at production levels of 2,000, 5,000, and 7,000, the respective approximate changes in revenue per unit change in production are \$6, \$0, and −\$4. That is, at the 2,000 output level, revenue increases as production increases; at the 5,000 output level, revenue does not change with a "small" change in production; and at the 7,000 output level, revenue decreases with an increase in production.

SOLUTION

(E) Graphing $R(x)$ and $C(x)$ in the same coordinate system results in Figure 3 on page 582. The intersection points are called the **break-even points**, because revenue equals cost at these production levels. The company neither makes nor loses money, but just breaks even. The break-even points are obtained as follows:

$$\begin{aligned}
C(x) &= R(x) \\
7{,}000 + 2x &= 10x - 0.001x^2 \\
0.001x^2 - 8x + 7{,}000 &= 0 \qquad \text{Solve by the quadratic formula} \\
x^2 - 8{,}000x + 7{,}000{,}000 &= 0 \qquad \text{(see Appendix A.7).} \\
x &= \frac{8{,}000 \pm \sqrt{8{,}000^2 - 4(7{,}000{,}000)}}{2} \\
&= \frac{8{,}000 \pm \sqrt{36{,}000{,}000}}{2} \\
&= \frac{8{,}000 \pm 6{,}000}{2} \\
&= 1{,}000, \quad 7{,}000
\end{aligned}$$

$$\begin{aligned}
R(1{,}000) &= 10(1{,}000) - 0.001(1{,}000)^2 = 9{,}000 \\
C(1{,}000) &= 7{,}000 + 2(1{,}000) = 9{,}000 \\
R(7{,}000) &= 10(7{,}000) - 0.001(7{,}000)^2 = 21{,}000 \\
C(7{,}000) &= 7{,}000 + 2(7{,}000) = 21{,}000
\end{aligned}$$

The break-even points are (1,000, 9,000) and (7,000, 21,000), as shown in Figure 3. Further examination of the figure shows that cost is greater than revenue for production levels between 0 and 1,000 and also between 7,000 and 10,000. Consequently, the company incurs a loss at these levels. By contrast, for production levels between 1,000 and 7,000, revenue is greater than cost, and the company makes a profit.

SOLUTION

(F) The **profit function** is

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= (10x - 0.001x^2) - (7{,}000 + 2x) \\ &= -0.001x^2 + 8x - 7{,}000 \end{aligned}$$

The domain of the cost function is $x \geq 0$, and the domain of the revenue function is $0 \leq x \leq 10{,}000$. The domain of the profit function is the set of x values for which both functions are defined—that is, $0 \leq x \leq 10{,}000$. The graph of the profit function is shown in Figure 4 on page 582. Notice that the x coordinates of the break-even points in Figure 3 are the x intercepts of the profit function. Furthermore, the intervals on which cost is greater than revenue and on which revenue is greater than cost correspond, respectively, to the intervals on which profit is negative and on which profit is positive.

SOLUTION

(G) The **marginal profit** is

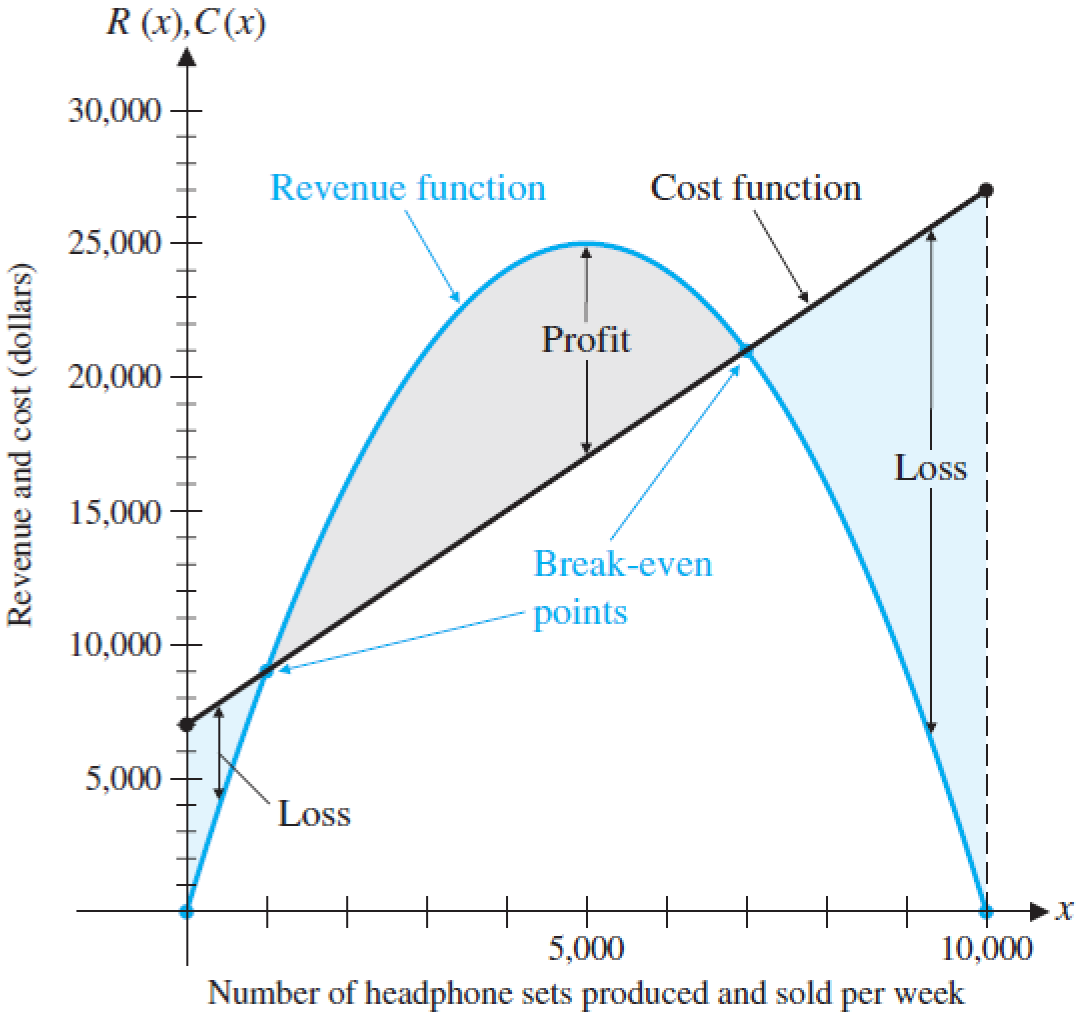
$$P'(x) = -0.002x + 8$$

For production levels of 1,000, 4,000, and 6,000, we have

$$P'(1{,}000) = 6 \qquad P'(4{,}000) = 0 \qquad P'(6{,}000) = -4$$

This means that at production levels of 1,000, 4,000, and 6,000, the respective approximate changes in profit per unit change in production are \$6, \$0, and −\$4. That is, at the 1,000 output level, profit will be increased if production is increased; at the 4,000 output level, profit does not change for "small" changes in production; and at the 6,000 output level, profits will decrease if production is increased. It seems that the best production level to produce a maximum profit is 4,000.

SOLUTION



R (x), C (x)
30,000
25,000
20,000
15,000
10,000
5,000
Revenue and cost (dollars)
Revenue function
Cost function
Profit
Loss
Break-even points
Loss
x
5,000
10,000
Number of headphone sets produced and sold per week

Marginal Average Cost, Revenue, and Profit

DEFINITION Marginal Average Cost, Revenue, and Profit

If x is the number of units of a product produced in some time interval, then

Cost per unit: **average cost** $= \overline{C}(x) = \frac{C(x)}{x}$

marginal average cost $= \overline{C}'(x) = \frac{d}{dx}\overline{C}(x)$

Revenue per unit: **average revenue** $= \overline{R}(x) = \frac{R(x)}{x}$

marginal average revenue $= \overline{R}'(x) = \frac{d}{dx}\overline{R}(x)$

Profit per unit: **average profit** $= \overline{P}(x) = \frac{P(x)}{x}$

marginal average profit $= \overline{P}'(x) = \frac{d}{dx}\overline{P}(x)$

Marginal Average Cost, Revenue, and Profit: Example

EXAMPLE Cost Analysis A small machine shop manufactures drill bits used in the petroleum industry. The manager estimates that the total daily cost (in dollars) of producing x bits is

$$C(x) = 1{,}000 + 25x - 0.1x^2$$

(A) Find $\overline{C}(x)$ and $\overline{C}'(x)$.

(B) Find $\overline{C}(10)$ and $\overline{C}'(10)$. Interpret these quantities.

(C) Use the results in part (B) to estimate the average cost per bit at a production level of 11 bits per day.

SOLUTION

(A)
$$\overline{C}(x) = \frac{C(x)}{x} = \frac{1{,}000 + 25x - 0.1x^2}{x}$$

$$= \frac{1{,}000}{x} + 25 - 0.1x \qquad \text{Average cost function}$$

$$\overline{C}'(x) = \frac{d}{dx}\overline{C}(x) = -\frac{1{,}000}{x^2} - 0.1 \qquad \text{Marginal average cost function}$$

Marginal Average Cost, Revenue, and Profit: Example

EXAMPLE Cost Analysis A small machine shop manufactures drill bits used in the petroleum industry. The manager estimates that the total daily cost (in dollars) of producing x bits is

$$C(x) = 1{,}000 + 25x - 0.1x^2$$

(A) Find $\overline{C}(x)$ and $\overline{C}'(x)$.

(B) Find $\overline{C}(10)$ and $\overline{C}'(10)$. Interpret these quantities.

(C) Use the results in part (B) to estimate the average cost per bit at a production level of 11 bits per day.

SOLUTION

(B) $$\overline{C}(10) = \frac{1{,}000}{10} + 25 - 0.1(10) = \$124$$

$$\overline{C}'(10) = -\frac{1{,}000}{10^2} - 0.1 = -\$10.10$$

At a production level of 10 bits per day, the average cost of producing a bit is \$124. This cost is decreasing at the rate of \$10.10 per bit.

(C) If production is increased by 1 bit, then the average cost per bit will decrease by approximately \$10.10. So, the average cost per bit at a production level of 11 bits per day is approximately $\$124 - \$10.10 = \$113.90$.