MAT101 – Mathematics-I

Quadratic Functions

Quadratic Functions

If the degree of a linear function is increased by one, we obtain a *second-degree function,* usually called a *quadratic function,* another basic function that we will need in our library of elementary functions. We will investigate relationships between quadratic functions and the solutions to quadratic equations and inequalities.

Indicate how the graph of each function is related to the graph of the function $h(x) = x^2$. Find the highest or lowest point, whichever exists, on each graph.

(A) $f(x) = (x - 3)^2 - 7 = x^2 - 6x + 2$

(B) $g(x) = 0.5(x + 2)^2 + 3 = 0.5x^2 + 2x + 5$

(C) $m(x) = -(x - 4)^2 + 8 = -x^2 + 8x - 8$

(D) $n(x) = -3(x + 1)^2 - 1 = -3x^2 - 6x - 4$

These figures are called *parabolas.** The functions that produce these parabolas are examples of the important class of *quadratic functions.*

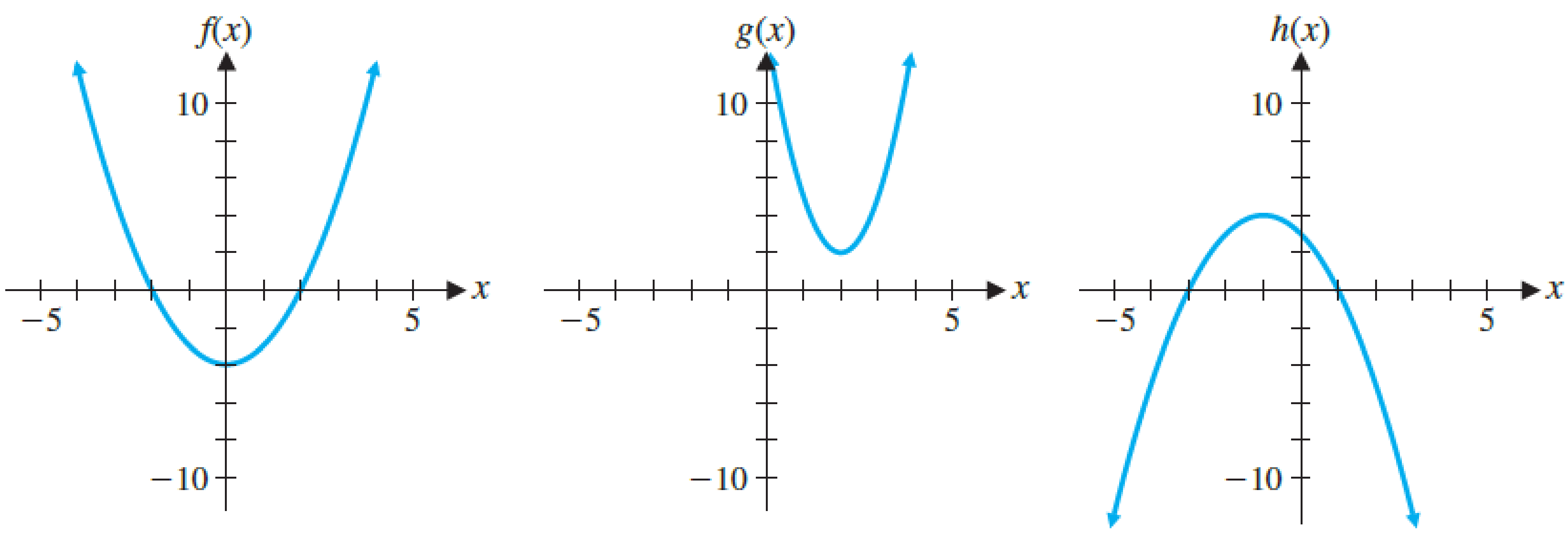
Quadratic Functions

DEFINITION Quadratic Functions

If a, b, and c are real numbers with $a \neq 0$, then the function

$$f(x) = ax^2 + bx + c \quad \text{Standard form}$$

is a **quadratic function** and its graph is a **parabola**.



(A) $f(x) = x^2 - 4$ (B) $g(x) = 3x^2 - 12x + 14$ (C) $h(x) = 3 - 2x - x^2$

Figure 2 **Graphs of quadratic functions**

Quadratic Functions

An x intercept of a function is also called a **zero** of the function. The x intercept of a linear function can be found by solving the linear equation $y = mx + b = 0$ for x, $m \neq 0$ (see Section 1.2). Similarly, the x intercepts of a quadratic function can be found by solving the quadratic equation $y = ax^2 + bx + c = 0$ for x, $a \neq 0$. Several methods for solving quadratic equations are discussed in Appendix A, Section A.7. The most popular of these is the **quadratic formula**.

If $ax^2 + bx + c = 0$, $a \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ provided } b^2 - 4ac \geq 0$$

Quadratic Functions

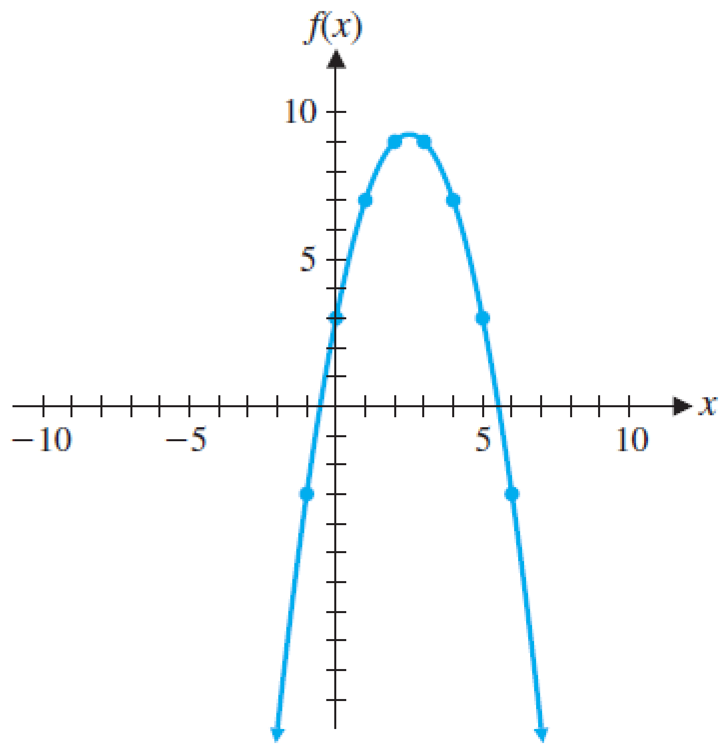
EXAMPLE 1 Intercepts, Equations, and Inequalities

(A) Sketch a graph of $f(x) = -x^2 + 5x + 3$ in a rectangular coordinate system.

SOLUTION

(A) Hand-sketching a graph of f:

x	y
-1	-3
0	3
1	7
2	9
3	9
4	7
5	3
6	-3



Quadratic Functions

(B) Find x and y intercepts algebraically to four decimal places.

SOLUTION

(B) Finding intercepts algebraically:

y intercept: $f(0) = -(0)^2 + 5(0) + 3 = 3$

x intercepts: $f(x) = 0$

$$-x^2 + 5x + 3 = 0$$ Quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic formula (see

$$x = \frac{-(5) \pm \sqrt{5^2 - 4(-1)(3)}}{2(-1)}$$

$$= \frac{-5 \pm \sqrt{37}}{-2} = -0.5414 \quad \text{or} \quad 5.5414$$

Quadratic Functions

Properties of Quadratic Functions and Their Graphs

Many useful properties of the quadratic function can be uncovered by transforming

$$f(x) = ax^2 + bx + c \quad a \neq 0$$

into the **vertex form**

$$f(x) = a(x - h)^2 + k$$

The process of *completing the square* (see Appendix A.7) is central to the transformation. We illustrate the process through a specific example and then generalize the results.

Quadratic Functions

Consider the quadratic function given by

$$f(x) = -2x^2 + 16x - 24 \tag{1}$$

We use completing the square to transform this function into vertex form:

$$\begin{aligned} f(x) &= -2x^2 + 16x - 24 \\ &= -2(x^2 - 8x) - 24 \\ &= -2(x^2 - 8x + \mathbf{?}) - 24 \end{aligned}$$

Factor the coefficient of x^2 out of the first two terms.

$$= -2(x^2 - 8x + \mathbf{16}) - 24 + \mathbf{32}$$

Add 16 to complete the square inside the parentheses. Because of the -2 outside the parentheses, we have actually added -32, so we must add 32 to the outside.

$$= -2(x - 4)^2 + 8$$

The transformation is complete and can be checked by multiplying out.

Therefore,

$$f(x) = -2(x - 4)^2 + 8 \tag{2}$$

Quadratic Functions

$$f(x) = -2(x - 4)^2 + 8$$

If $x = 4$, then $-2(x - 4)^2 = 0$ and $f(4) = 8$. For any other value of x, the negative number $-2(x - 4)^2$ is added to 8, making it smaller. Therefore,

$$f(4) = 8$$

is the *maximum value* of $f(x)$ for all x. Furthermore, if we choose any two x values that are the same distance from 4, we will obtain the same function value. For example, $x = 3$ and $x = 5$ are each one unit from $x = 4$ and their function values are

$$f(3) = -2(3 - 4)^2 + 8 = 6$$
$$f(5) = -2(5 - 4)^2 + 8 = 6$$

Therefore, the vertical line $x = 4$ is a line of symmetry. That is, if the graph of equation (1) is drawn on a piece of paper and the paper is folded along the line $x = 4$, then the two sides of the parabola will match exactly. All these results are illustrated by graphing equations (1) and (2) and the line $x = 4$ simultaneously in the same coordinate system (Fig. 3).

Quadratic Functions

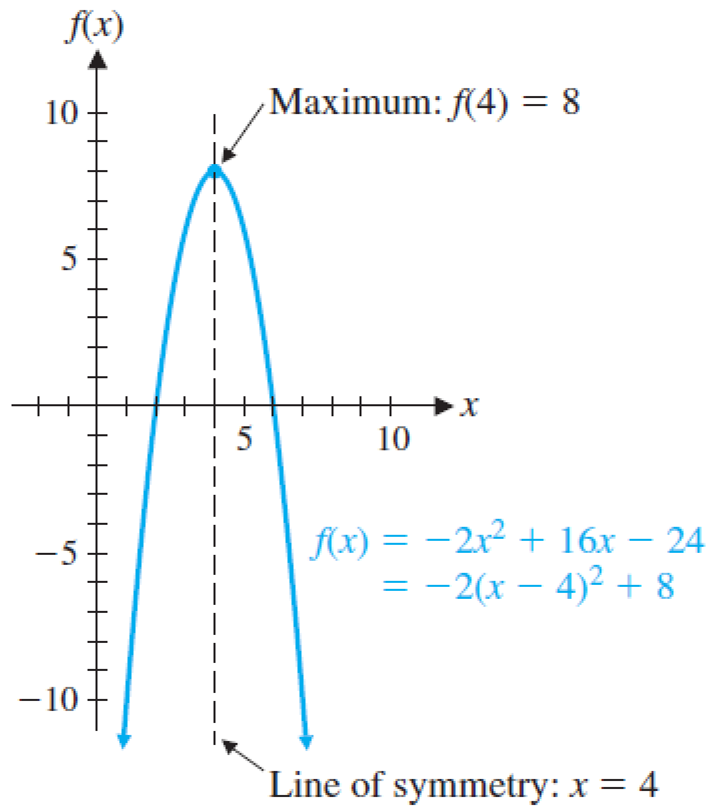


Figure 3 **Graph of a quadratic function**

Quadratic Functions

From the preceding discussion, we see that as x moves from left to right, $f(x)$ is increasing on $(-\infty, 4]$, and decreasing on $[4, \infty)$, and that $f(x)$ can assume no value greater than 8. Thus,

$$\text{Range of } f: \quad y \leq 8 \quad \text{or} \quad (-\infty, 8]$$

In general, the graph of a quadratic function is a parabola with line of symmetry parallel to the vertical axis. The lowest or highest point on the parabola, whichever exists, is called the **vertex**. The maximum or minimum value of a quadratic function always occurs at the vertex of the parabola. The line of symmetry through the vertex is called the **axis** of the parabola. In the example above, $x = 4$ is the axis of the parabola and $(4, 8)$ is its vertex.

Applying the graph transformation properties discussed in Section 2.2 to the transformed equation,

$$f(x) = -2x^2 + 16x - 24$$
$$= -2(x-4)^2 + 8$$

we see that the graph of $f(x) = -2x^2 + 16x - 24$ is the graph of $g(x) = x^2$ ver cally stretched by a factor of 2, reflected in the x axis, and shifted to the right 4 uni and up 8 units, as shown in Figure 4.

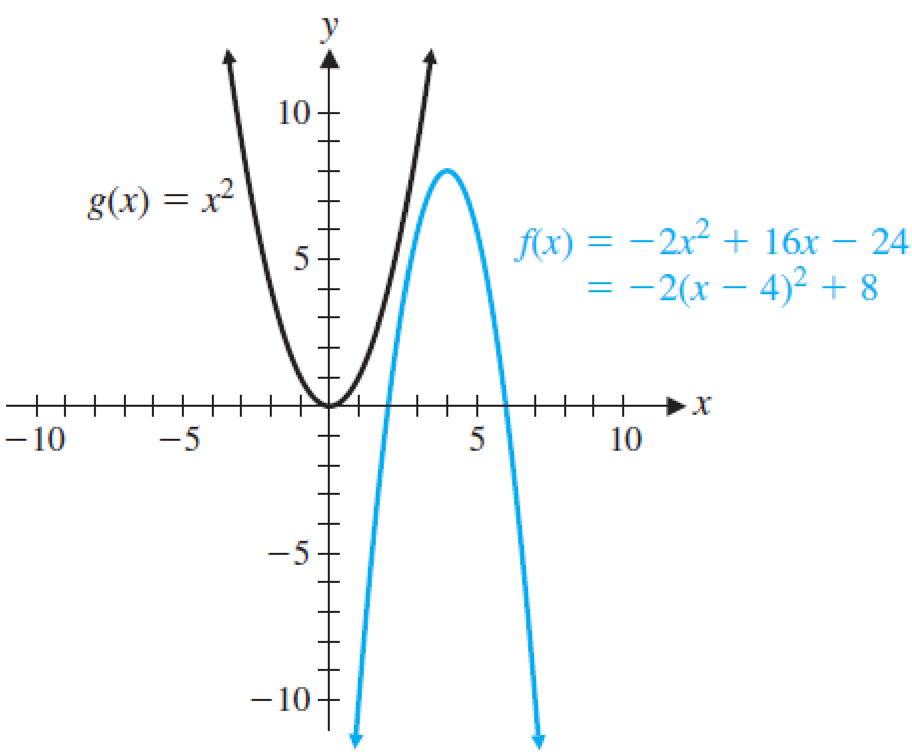


Figure 4 **Graph of *f* is the graph of *g* transformed**

Quadratic Functions

Note the important results we have obtained from the vertex form of the quadratic function f:

- The vertex of the parabola
- The axis of the parabola
- The maximum value of $f(x)$
- The range of the function f
- The relationship between the graph of $g(x) = x^2$ and the graph of $f(x) = -2x^2 + 16x - 24$

The preceding discussion is generalized to all quadratic functions in the following summary:

Quadratic Functions

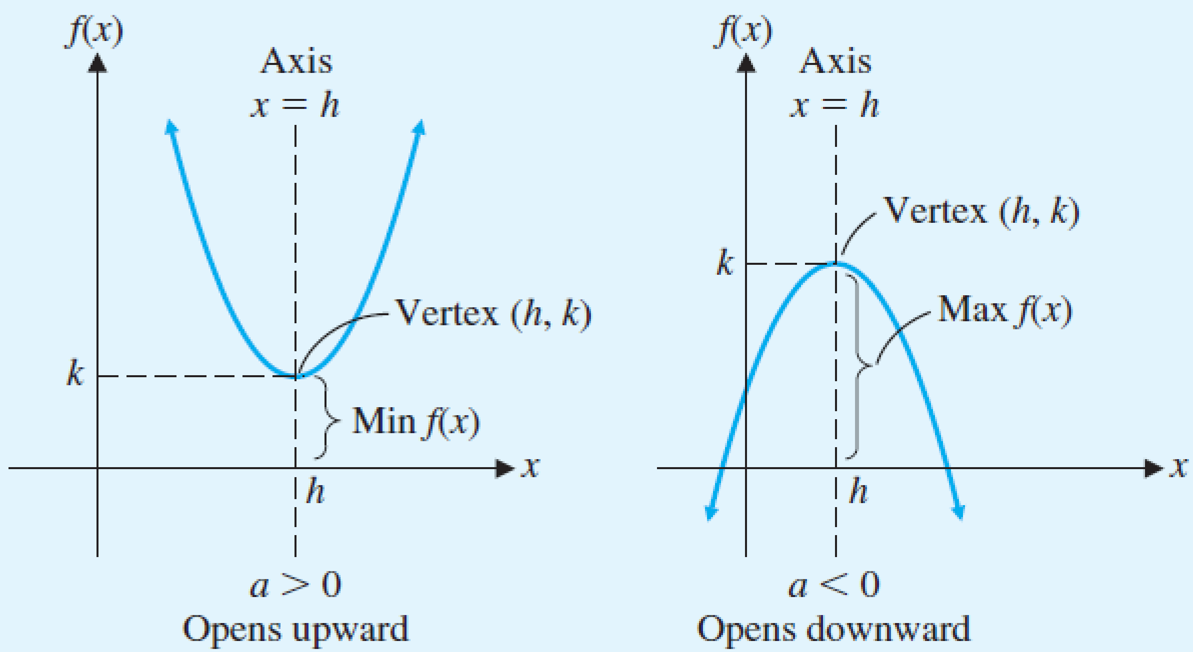
SUMMARY Properties of a Quadratic Function and Its Graph

Given a quadratic function and the vertex form obtained by completing the square

$$f(x) = ax^2 + bx + c \qquad a \neq 0 \qquad \text{Standard form}$$
$$= a(x - h)^2 + k \qquad \text{Vertex form}$$

we summarize general properties as follows:

1. The graph of f is a parabola:



Quadratic Functions

2. Vertex: (h, k) (parabola increases on one side of the vertex and decreases on the other)

3. Axis (of symmetry): $x = h$ (parallel to y axis)

4. $f(h) = k$ is the minimum if $a > 0$ and the maximum if $a < 0$

5. Domain: All real numbers. Range: $(-\infty, k]$ if $a < 0$ or $[k, \infty)$ if $a > 0$

6. The graph of f is the graph of $g(x) = ax^2$ translated horizontally h units and vertically k units.

EXAMPLE 2 Analyzing a Quadratic Function Given the quadratic function

$$f(x) = 0.5x^2 - 6x + 21$$

(A) Find the vertex form for f.

SOLUTION

(A) Complete the square to find the vertex form:

$$\begin{aligned} f(x) &= 0.5x^2 - 6x + 21 \\ &= 0.5(x^2 - 12x + ?) + 21 \\ &= 0.5(x^2 - 12x + 36) + 21 - 18 \\ &= 0.5(x - 6)^2 + 3 \end{aligned}$$

Quadratic Functions

EXAMPLE 2 Analyzing a Quadratic Function Given the quadratic function

$$f(x) = 0.5x^2 - 6x + 21$$

(B) Find the vertex and the maximum or minimum. State the range of f.

(C) Describe how the graph of function f can be obtained from the graph of $g(x) = x^2$ using transformations.

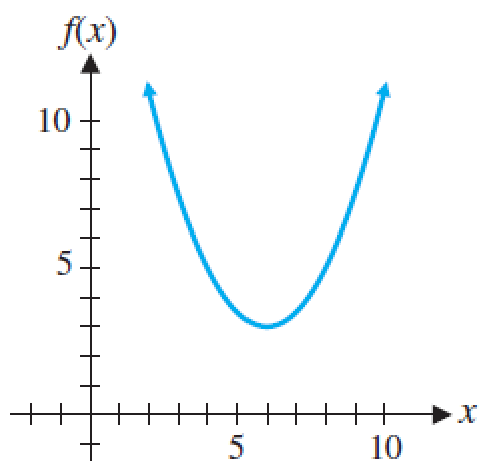
(D) Sketch a graph of function f in a rectangular coordinate system.

SOLUTION

(B) From the vertex form, we see that $h = 6$ and $k = 3$. Thus, vertex: $(6, 3)$; minimum: $f(6) = 3$; range: $y \geq 3$ or $[3, \infty)$.

(C) The graph of $f(x) = 0.5(x - 6)^2 + 3$ is the same as the graph of $g(x) = x^2$ vertically shrunk by a factor of 0.5, and shifted to the right 6 units and up 3 units.

(D)



Quadratic Functions

Matched Problem 2 Given the quadratic function $f(x) = -0.25x^2 - 2x + 2$

(A) Find the vertex form for f.

(B) Find the vertex and the maximum or minimum. State the range of f.

(C) Describe how the graph of function f can be obtained from the graph of $g(x) = x^2$ using transformations.

(D) Sketch a graph of function f in a rectangular coordinate system.

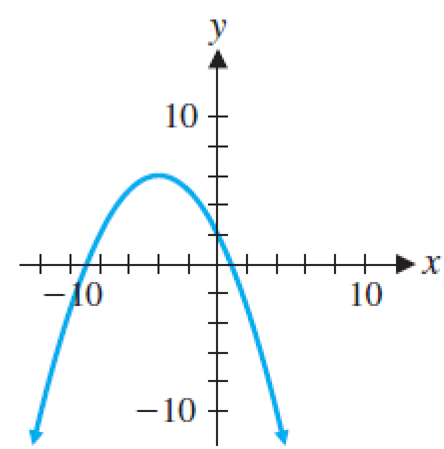
Answers to Matched Problem

2. (A) $f(x) = -0.25(x + 4)^2 + 6$.

(B) Vertex: $(-4, 6)$; maximum: $f(-4) = 6$; range: $y \leq 6$ or $(-\infty, 6]$

(C) The graph of $f(x) = -0.25(x + 4)^2 + 6$ is the same as the graph of $g(x) = x^2$ vertically shrunk by a factor of 0.25, reflected in the x axis, and shifted 4 units to the left and 6 units up.

(D)



Quadratic Functions: Applications

EXAMPLE 3 Maximum Revenue This is a continuation of Example 7 in Section 2.1. Recall that the financial department in the company that produces a digital camera arrived at the following price–demand function and the corresponding revenue function:

$$p(x) = 94.8 - 5x \qquad \text{Price–demand function}$$

$$R(x) = xp(x) = x(94.8 - 5x) \qquad \text{Revenue function}$$

where $p(x)$ is the wholesale price per camera at which x million cameras can be sold and $R(x)$ is the corresponding revenue (in millions of dollars). Both functions have domain $1 \le x \le 15$.

(A) Find the value of x to the nearest thousand cameras that will generate the maximum revenue. What is the maximum revenue to the nearest thousand dollars? Solve the problem algebraically by completing the square.

(B) What is the wholesale price per camera (to the nearest dollar) that generates the maximum revenue?

Quadratic Functions: Applications

SOLUTION

(A) Algebraic solution:

$$\begin{aligned} R(x) &= x(94.8 - 5x) \\ &= -5x^2 + 94.8x \\ &= -5(x^2 - 18.96x + ?) \\ &= -5(x^2 - 18.96x + 89.8704) + 449.352 \\ &= -5(x - 9.48)^2 + 449.352 \end{aligned}$$

The maximum revenue of 449.352 million dollars ($449,352,000) occurs when $x = 9.480$ million cameras (9,480,000 cameras).

(B) Finding the wholesale price per camera: Use the price-demand function for an output of 9.480 million cameras:

$$\begin{aligned} p(x) &= 94.8 - 5x \\ p(9.480) &= 94.8 - 5(9.480) \\ &= \$47 \text{ per camera} \end{aligned}$$

Quadratic Functions: Applications

EXAMPLE 4 Break-Even Analysis Use the revenue function from Example 3 and the cost function from Matched Problem 3:

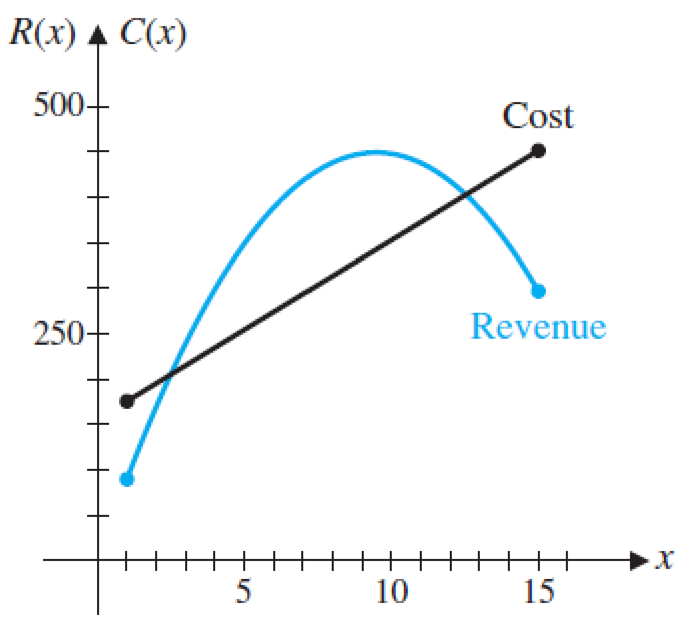
$$R(x) = x(94.8 - 5x) \quad \text{Revenue function}$$
$$C(x) = 156 + 19.7x \quad \text{Cost function}$$

Both have domain $1 \le x \le 15$.

(A) Sketch the graphs of both functions in the same coordinate system.

(B) **Break-even points** are the production levels at which $R(x) = C(x)$. Find the break-even points algebraically to the nearest thousand cameras.

SOLUTION (A) Sketch of functions:



Quadratic Functions: Applications

SOLUTION

(B) Algebraic solution:

Find x such that $R(x) = C(x)$:

$$x(94.8 - 5x) = 156 + 19.7x$$

$$-5x^2 + 75.1x - 156 = 0$$

$$x = \frac{-75.1 \pm \sqrt{75.1^2 - 4(-5)(-156)}}{2(-5)}$$

$$= \frac{-75.1 \pm \sqrt{2{,}520.01}}{-10}$$

$$x = 2.490 \quad \text{and} \quad 12.530$$

The company breaks even at $x = 2.490$ million cameras (2,490,000 cameras) and at $x = 12.530$ million cameras (12,530,000 cameras).

Quadratic Functions

Matched Problem 4 Use the profit equation from Matched Problem 3:

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= -5x^2 + 75.1x - 156 \quad \text{Profit function} \end{aligned}$$

$$\text{Domain:} \quad 1 \le x \le 15$$

(A) Sketch a graph of the profit function in a rectangular coordinate system.

(B) Break-even points occur when $P(x) = 0$. Find the break-even points algebraically to the nearest thousand cameras.

Polynomial and Rational Functions

Polynomial and Rational Functions

Linear and quadratic functions are special cases of the more general class of *polynomial functions*. Polynomial functions are a special case of an even larger class of functions, the *rational functions*. We will describe the basic features of the graphs of polynomial and rational functions. We will use these functions to solve real-world problems where linear or quadratic models are inadequate; for example, to determine the relationship between length and weight of a species of fish, or to model the training of new employees.

Polynomial Functions

A linear function has the form $f(x) = mx + b$ (where $m \neq 0$) and is a polynomial function of degree 1. A quadratic function has the form $f(x) = ax^2 + bx + c$ (where $a \neq 0$) and is a polynomial function of degree 2. Here is the general definition of a polynomial function.

DEFINITION Polynomial Function

A **polynomial function** is a function that can be written in the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

for n a nonnegative integer, called the **degree** of the polynomial. The coefficients $a_0, a_1, \ldots, a_n$ are real numbers with $a_n \neq 0$. The **domain** of a polynomial function is the set of all real numbers.

Figure 1 shows graphs of representative polynomial functions of degrees 1 through 6. The figure, which also appears on the inside back cover, suggests some general properties of graphs of polynomial functions.

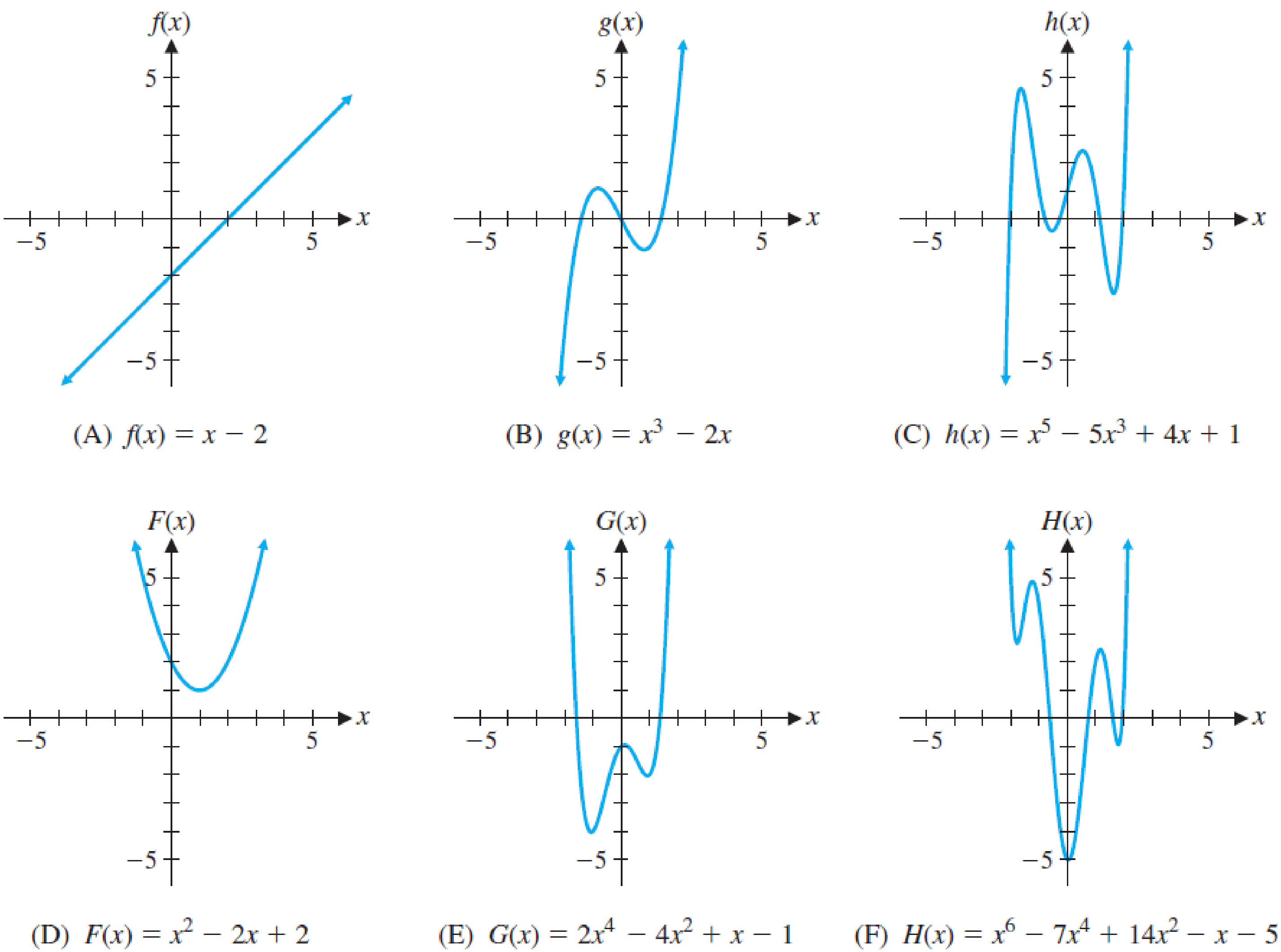


Figure 1 **Graphs of polynomial functions**

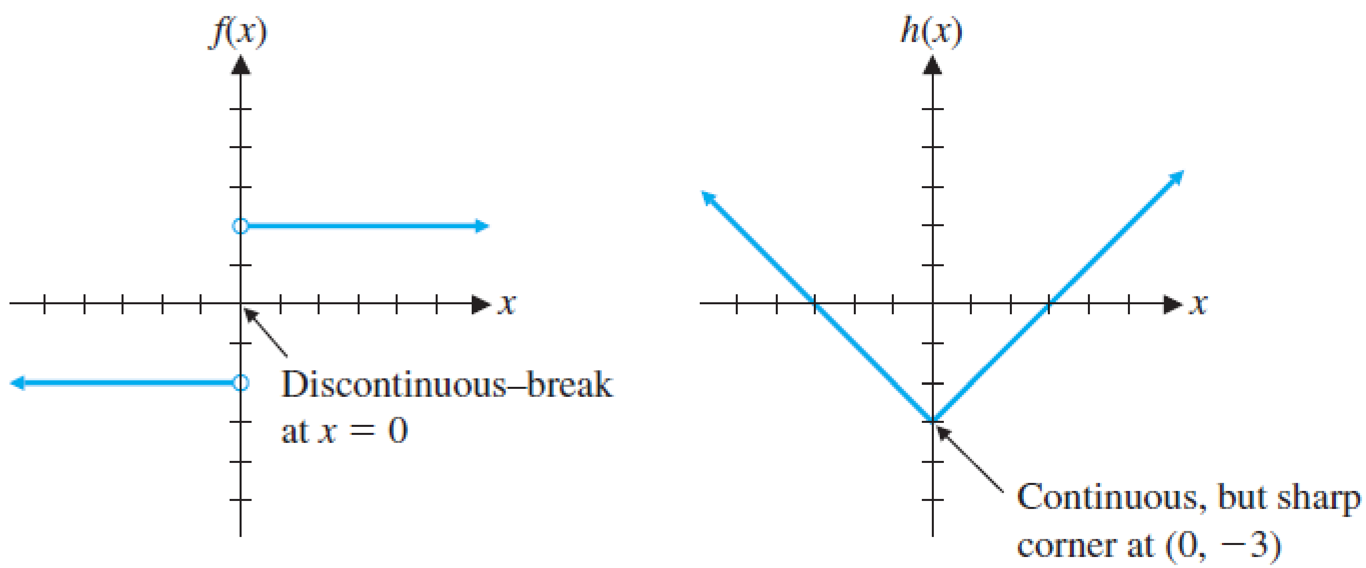
Polynomial and Rational Functions

Notice that the odd-degree polynomial graphs start negative, end positive, and cross the x axis at least once. The even-degree polynomial graphs start positive, end positive, and may not cross the x axis at all. In all cases in Figure 1, the **leading coefficient**—that is, the coefficient of the highest-degree term—was chosen positive. If any leading coefficient had been chosen negative, then we would have a similar graph but reflected in the x axis.

A polynomial of degree n can have, at most, n linear factors. Therefore, the graph of a polynomial function of positive degree n can intersect the x axis at most n times. Note from Figure 1 that a polynomial of degree n may intersect the x axis fewer than n times. An x intercept of a function is also called a **zero*** or **root** of the function.

Polynomial and Rational Functions

The graph of a polynomial function is **continuous,** with no holes or breaks. That is, the graph can be drawn without removing a pen from the paper. Also, the graph of a polynomial has no sharp corners. Figure 2 shows the graphs of two functions—one that is not continuous, and the other that is continuous but with a sharp corner. Neither function is a polynomial.



(A) $f(x) = \dfrac{2|x|}{x}$ (B) $h(x) = |x| - 3$

Figure 2 **Discontinuous and sharp-corner functions**

Polynomial and Rational Functions

Rational Functions

Just as rational numbers are defined in terms of quotients of integers, *rational functions* are defined in terms of quotients of polynomials. The following equations specify rational functions:

$$f(x) = \frac{1}{x} \quad g(x) = \frac{x-2}{x^2 - x - 6} \quad h(x) = \frac{x^3 - 8}{x}$$

$$p(x) = 3x^2 - 5x \quad q(x) = 7 \quad r(x) = 0$$

DEFINITION Rational Function

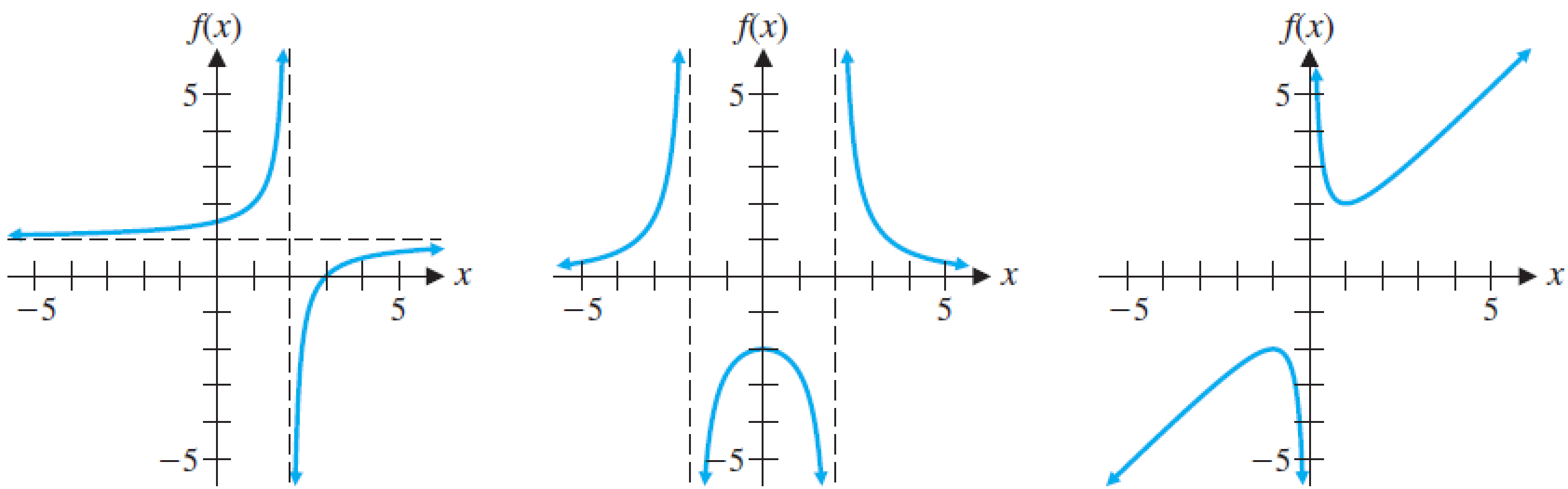
A **rational function** is any function that can be written in the form

$$f(x) = \frac{n(x)}{d(x)} \quad d(x) \neq 0$$

where $n(x)$ and $d(x)$ are polynomials. The **domain** is the set of all real numbers such that $d(x) \neq 0$.

Polynomial and Rational Functions

Figure 4 shows the graphs of representative rational functions. Note, for example that in Figure 4A the line $x = 2$ is a *vertical asymptote* for the function. The graph of f gets closer to this line as x gets closer to 2. The line $y = 1$ in Figure 4A is a *horizontal asymptote* for the function. The graph of f gets closer to this line as x increases or decreases without bound.



(A) $f(x) = \dfrac{x - 3}{x - 2}$ (B) $f(x) = \dfrac{8}{x^2 - 4}$ (C) $f(x) = x + \dfrac{1}{x}$

Figure 4 **Graphs of rational functions**

Polynomial and Rational Functions

EXAMPLE 2 Graphing Rational Functions Given the rational function:

$$f(x) = \frac{3x}{x^2 - 4}$$

(A) Find the domain.

(B) Find the x and y intercepts.

(C) Find the equations of all vertical asymptotes.

(D) If there is a horizontal asymptote, find its equation.

SOLUTION

(A) $x^2 - 4 = (x - 2)(x + 2)$, so the denominator is 0 if $x = -2$ or $x = 2$. Therefore the domain is the set of all real numbers except -2 and 2.

(B) *x intercepts:* $f(x) = 0$ only if $3x = 0$, or $x = 0$. So the only x intercept is 0.
y intercept:

$$f(0) = \frac{3 \cdot 0}{0^2 - 4} = \frac{0}{-4} = 0$$

So the y intercept is 0.

EXAMPLE 2 Graphing Rational Functions Given the rational function:

$$f(x) = \frac{3x}{x^2 - 4}$$

(A) Find the domain.

(B) Find the x and y intercepts.

(C) Find the equations of all vertical asymptotes.

(D) If there is a horizontal asymptote, find its equation.

SOLUTION

(C) Consider individually the values of x for which the denominator is 0, namely, 2 and -2, found in part (A).

(D) Rewrite $f(x)$ by dividing each term in the numerator and denominator by the highest power of x in $f(x)$.

$$f(x) = \frac{3x}{x^2 - 4} = \frac{\frac{3x}{x^2}}{\frac{x^2}{x^2} - \frac{4}{x^2}} = \frac{\frac{3}{x}}{1 - \frac{4}{x^2}}$$

As x increases or decreases without bound, the numerator tends to 0 and the denominator tends to 1; so, $f(x)$ tends to 0. The line $y = 0$ is a horizontal asymptote.

Polynomial and Rational Functions

Matched Problem 2 Given the rational function $g(x) = \dfrac{3x + 3}{x^2 - 9}$,

(A) Find the domain.

(B) Find the x and y intercepts.

(C) Find the equations of all vertical asymptotes.

(D) If there is a horizontal asymptote, find its equation.

PROCEDURE Vertical and Horizontal Asymptotes of Rational Functions

Consider the rational function

$$f(x) = \frac{n(x)}{d(x)}$$

where $n(x)$ and $d(x)$ are polynomials.

Vertical asymptotes:

Case 1. Suppose $n(x)$ and $d(x)$ have no real zero in common. If c is a real number such that $d(c) = 0$, then the line $x = c$ is a vertical asymptote of the graph of f.

Case 2. If $n(x)$ and $d(x)$ have one or more real zeros in common, cancel common linear factors, and apply Case 1 to the reduced function. (The reduced function has the same asymptotes as f.)

Horizontal asymptote:

Case 1. If degree $n(x) <$ degree $d(x)$, then $y = 0$ is the horizontal asymptote.

Case 2. If degree $n(x) =$ degree $d(x)$, then $y = a/b$ is the horizontal asymptote, where a is the leading coefficient of $n(x)$, and b is the leading coefficient of $d(x)$.

Case 3. If degree $n(x) >$ degree $d(x)$, there is no horizontal asymptote.

Polynomial and Rational Functions

EXAMPLE 3 Finding Asymptotes Find the vertical and horizontal asymptotes of the rational function

$$f(x) = \frac{3x^2 + 3x - 6}{2x^2 - 2}$$

SOLUTION Vertical asymptotes We factor the numerator $n(x)$ and the denominator $d(x)$:

$$n(x) = 3(x^2 + x - 2) = 3(x - 1)(x + 2)$$
$$d(x) = 2(x^2 - 1) = 2(x - 1)(x + 1)$$

The reduced function is

$$\frac{3(x + 2)}{2(x + 1)}$$

which, by the procedure, has the vertical asymptote $x = -1$. Therefore, $x = -1$ is the only vertical asymptote of f.

Horizontal asymptote Both $n(x)$ and $d(x)$ have degree 2 (Case 2 of the procedure for horizontal asymptotes). The leading coefficient of the numerator $n(x)$ is 3, and the leading coefficient of the denominator $d(x)$ is 2. So $y = 3/2$ is the horizontal asymptote.

Applications

57. **Average cost.** A company manufacturing snowboards has fixed costs of \$200 per day and total costs of \$3,800 per day at a daily output of 20 boards.

(A) Assuming that the total cost per day, $C(x)$, is linearly related to the total output per day, x, write an equation for the cost function.

(B) The average cost per board for an output of x boards is given by $\overline{C}(x) = C(x)/x$. Find the average cost function.

(C) Sketch a graph of the average cost function, including any asymptotes, for $1 \leq x \leq 30$.

(D) What does the average cost per board tend to as production increases?