MAT101 – Mathematics-I

Limits

Limits: A Graphical Approach

We introduce the important concept of a *limit* through an example, which leads to an intuitive definition of the concept.

EXAMPLE 2 Analyzing a Limit Let $f(x) = x + 2$. Discuss the behavior of the values of $f(x)$ when x is close to 2.

SOLUTION We begin by drawing a graph of f that includes the domain value $x = 2$ (Fig. 2).

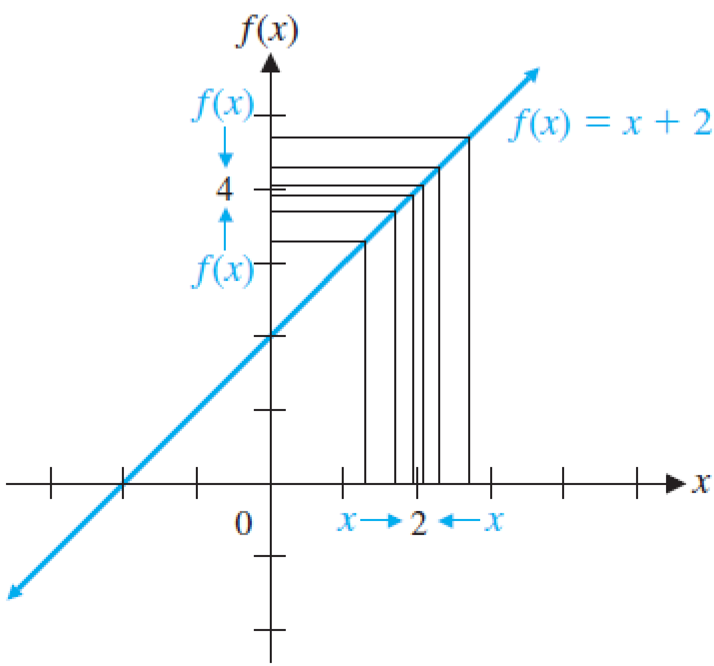


Figure 2

Limits: A Graphical Approach

In Figure 2, we are using a static drawing to describe a dynamic process. This requires careful interpretation. The thin vertical lines in Figure 2 represent values of x that are close to 2. The corresponding horizontal lines identify the value of $f(x)$ associated with each value of x. [Example 1 dealt with the relationship between x and $f(x)$ on a graph.] The graph in Figure 2 indicates that as the values of x get closer and closer to 2 on either side of 2, the corresponding values of $f(x)$ get closer and closer to 4. Symbolically, we write

$$\lim_{x \to 2} f(x) = 4$$

This equation is read as "The limit of $f(x)$ as x approaches 2 is 4." Note that $f(2) = 4$. That is, the value of the function at 2 and the limit of the function as x approaches 2 are the same. This relationship can be expressed as

$$\lim_{x \to 2} f(x) = f(2) = 4$$

Graphically, this means that there is no hole or break in the graph of f at $x = 2$.

Limits: A Graphical Approach

We now present an informal definition of the important concept of a limit.

DEFINITION Limit

We write

$$\lim_{x \to c} f(x) = L \qquad \text{or} \qquad f(x) \to L \quad \text{as} \quad x \to c$$

if the functional value $f(x)$ is close to the single real number L whenever x is close, but not equal, to c (on either side of c).

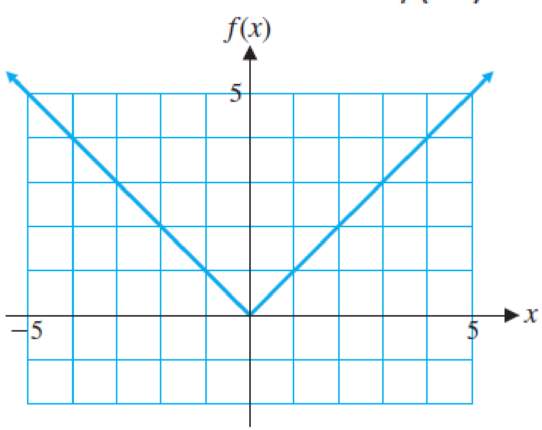
Note: The existence of a limit at c has nothing to do with the value of the function at c. In fact, c may not even be in the domain of f. However, the function must be defined on both sides of c.

The next example involves the **absolute value function**:

$$f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

$f(-2) = |-2| = -(-2) = 2$

$f(3) = |3| = 3$



Limits: A Graphical Approach

EXAMPLE 3 Analyzing a Limit Let $h(x) = |x|/x$. Explore the behavior of $h(x)$ for x near, but not equal, to 0. Find $\lim_{x\to 0} h(x)$ if it exists.

SOLUTION The function h is defined for all real numbers except 0 [$h(0) = |0|/0$ is undefined]. For example,

$$h(-2) = \frac{|-2|}{-2} = \frac{2}{-2} = -1$$

Note that if x is any negative number, then $h(x) = -1$ (if $x < 0$, then the numerator $|x|$ is positive but the denominator x is negative, so $h(x) = |x|/x = -1$). If x is any positive number, then $h(x) = 1$ (if $x > 0$, then the numerator $|x|$ is equal to the denominator x, so $h(x) = |x|/x = 1$). Figure 4 illustrates the behavior of $h(x)$ for x near 0. Note that the absence of a solid dot on the vertical axis indicates that h is not defined when $x = 0$.

When x is near 0 (on either side of 0), is $h(x)$ near one specific number? The answer is "No," because $h(x)$ is -1 for $x < 0$ and 1 for $x > 0$. Consequently, we say that

$$\lim_{x\to 0} \frac{|x|}{x} \text{ does not exist}$$

Neither $h(x)$ nor the limit of $h(x)$ exists at $x = 0$. However, the limit from the left and the limit from the right both exist at 0, but they are not equal.

Limits: A Graphical Approach

Matched Problem 3 Graph

$$h(x) = \frac{x - 2}{|x - 2|}$$

and find $\lim_{x \to 2} h(x)$ if it exists.

In Example 3, we see that the values of the function $h(x)$ approach two different numbers, depending on the direction of approach, and it is natural to refer to these values as "the limit from the left" and "the limit from the right." These experiences suggest that the notion of **one-sided limits** will be very useful in discussing basic limit concepts.

DEFINITION One-Sided Limits

We write

$$\lim_{x \to c^-} f(x) = K$$

$x \to c^-$ is read "x approaches c from the left" and means $x \to c$ and $x < c$.

and call K the **limit from the left** or the **left-hand limit** if $f(x)$ is close to K whenever x is close to, but to the left of, c on the real number line. We write

$$\lim_{x \to c^+} f(x) = L$$

$x \to c^+$ is read "x approaches c from the right" and means $x \to c$ and $x > c$.

and call L the **limit from the right** or the **right-hand limit** if $f(x)$ is close to L whenever x is close to, but to the right of, c on the real number line.

Limits: A Graphical Approach

THEOREM 1 On the Existence of a Limit

For a (two-sided) limit to exist, the limit from the left and the limit from the right must exist and be equal. That is,

$$\lim_{x \to c} f(x) = L \text{ if and only if } \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$$

In Example 3,

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1 \qquad \text{and} \qquad \lim_{x \to 0^+} \frac{|x|}{x} = 1$$

Since the left- and right-hand limits are *not* the same,

$$\lim_{x \to 0} \frac{|x|}{x} \text{ does not exist}$$

Limits: A Graphical Approach

EXAMPLE 4 Analyzing Limits Graphically Given the graph of the function f in Figure 5, discuss the behavior of $f(x)$ for x near (A) -1, (B) 1, and (C) 2.

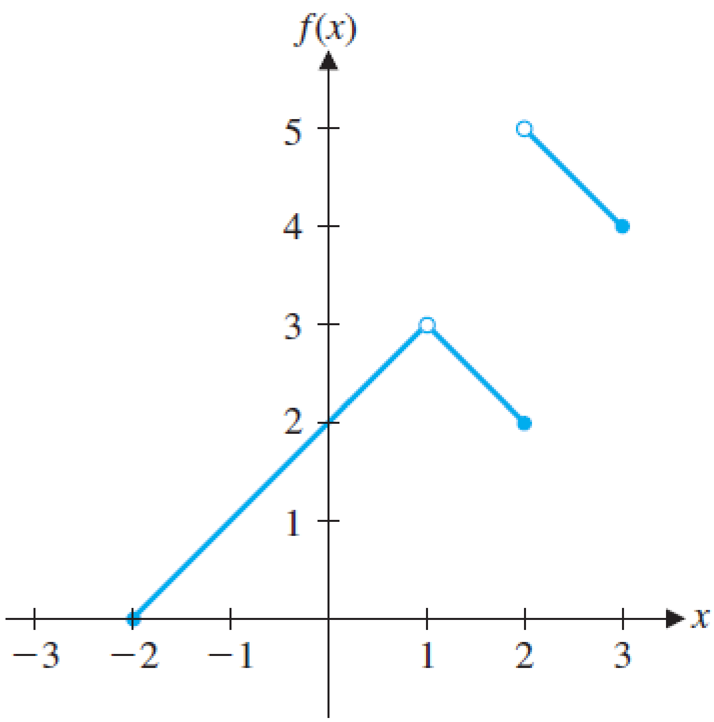
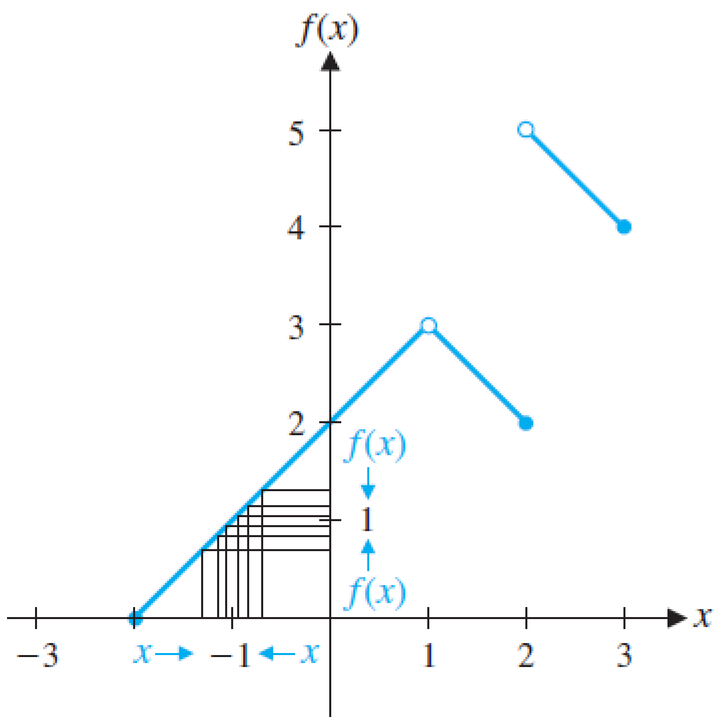


Figure 5

Limits: A Graphical Approach

SOLUTION

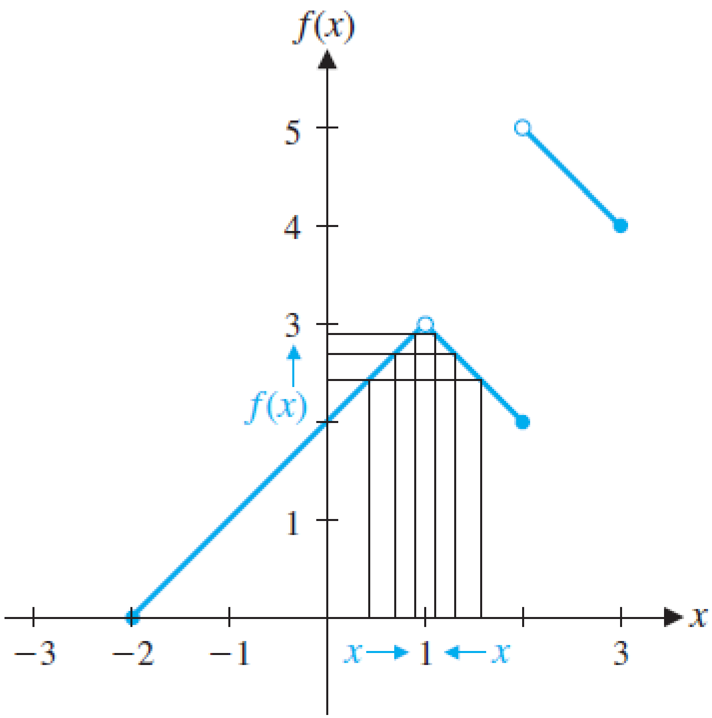
(A) Since we have only a graph to work with, we use vertical and horizontal lines to relate the values of x and the corresponding values of $f(x)$. For any x near -1 on either side of -1, we see that the corresponding value of $f(x)$, determined by a horizontal line, is close to 1.



$$\lim_{x \to -1^-} f(x) = 1$$
$$\lim_{x \to -1^+} f(x) = 1$$
$$\lim_{x \to -1} f(x) = 1$$
$$f(-1) = 1$$

Limits: A Graphical Approach

(B) Again, for any x near, but not equal to, 1, the vertical and horizontal lines indicate that the corresponding value of $f(x)$ is close to 3. The open dot at (1, 3), together with the absence of a solid dot anywhere on the vertical line through $x = 1$, indicates that $f(1)$ is not defined.



$$\lim_{x \to 1^-} f(x) = 3$$

$$\lim_{x \to 1^+} f(x) = 3$$

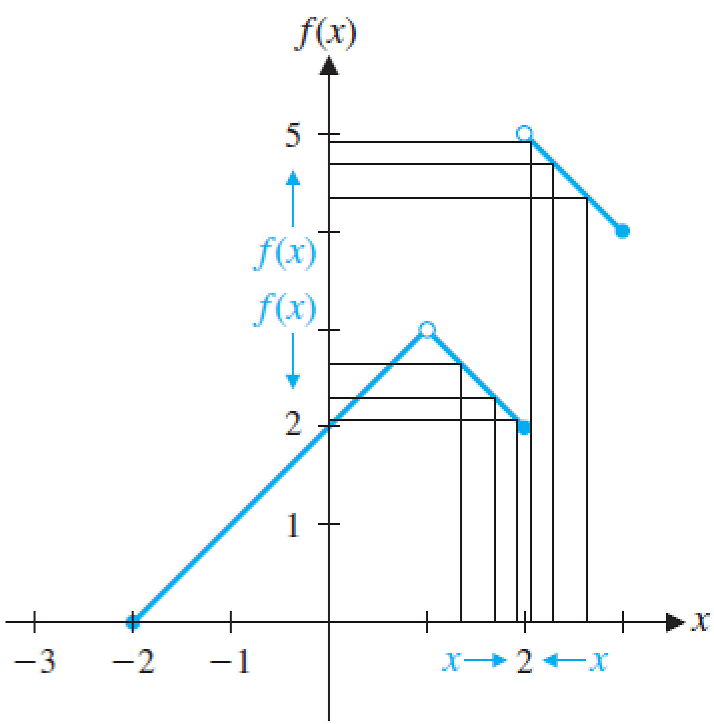
$$\lim_{x \to 1} f(x) = 3$$

$f(1)$ not defined

In Example 4B, note that $\lim_{x \to 1} f(x)$ exists even though f is not defined at $x = 1$ and the graph has a hole at $x = 1$. In general, the value of a function at $x = c$ has no effect on the limit of the function as x approaches c.

Limits: A Graphical Approach

(C) The abrupt break in the graph at $x = 2$ indicates that the behavior of the graph near $x = 2$ is more complicated than in the two preceding cases. If x is close to 2 on the left side of 2, the corresponding horizontal line intersects the y axis at a point close to 2. If x is close to 2 on the right side of 2, the corresponding horizontal line intersects the y axis at a point close to 5. This is a case where the one-sided limits are different.



$$\lim_{x \to 2^-} f(x) = 2$$

$$\lim_{x \to 2^+} f(x) = 5$$

$$\lim_{x \to 2} f(x) \text{ does not exist}$$

$$f(2) = 2$$

Limits: A Graphical Approach

Matched Problem 4 Given the graph of the function f shown in Figure 6, discuss the following, as we did in Example 4:

(A) Behavior of $f(x)$ for x near 0

(B) Behavior of $f(x)$ for x near 1

(C) Behavior of $f(x)$ for x near 3

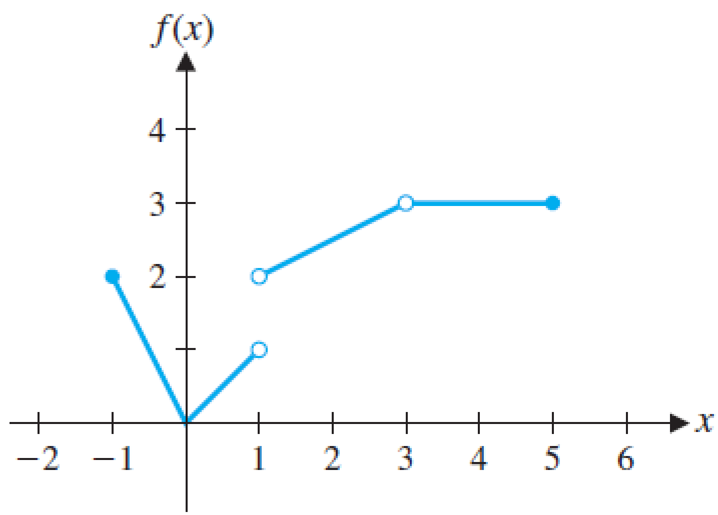


Figure 6

Limits: An Algebraic Approach

THEOREM 2 Properties of Limits

Let f and g be two functions, and assume that

$$\lim_{x \to c} f(x) = L \qquad \lim_{x \to c} g(x) = M$$

where L and M are real numbers (both limits exist). Then

1. $\lim_{x \to c} k = k$ for any constant k
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = L + M$
4. $\lim_{x \to c} [f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x) = L - M$
5. $\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x) = kL$ for any constant k
6. $\lim_{x \to c} [f(x) \cdot g(x)] = [\lim_{x \to c} f(x)][\lim_{x \to c} g(x)] = LM$
7. $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \dfrac{L}{M}$ if $M \neq 0$
8. $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)} = \sqrt[n]{L}$ $L > 0$ for n even

Limits: An Algebraic Approach

EXAMPLE 5 Using Limit Properties Find $\lim_{x \to 3} (x^2 - 4x)$.

SOLUTION

$$\lim_{x \to 3} (x^2 - 4x) = \lim_{x \to 3} x^2 - \lim_{x \to 3} 4x \qquad \text{Property 4}$$

$$= \left(\lim_{x \to 3} x\right) \cdot \left(\lim_{x \to 3} x\right) - 4 \lim_{x \to 3} x \qquad \text{Properties 5 and 6}$$

$$= \left(\lim_{x \to 3} x\right)^2 - 4 \lim_{x \to 3} x \qquad \text{Definition of exponent}$$

$$= 3^2 - 4 \cdot 3 = -3$$

So, omitting the steps in the dashed boxes,

$$\lim_{x \to 3} (x^2 - 4x) = 3^2 - 4 \cdot 3 = -3$$

Matched Problem 5 Find $\lim_{x \to -2} (x^2 + 5x)$.

Limits: An Algebraic Approach

If $f(x) = x^2 - 4$ and c is any real number, then, just as in Example 5

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x^2 - 4x) = c^2 - 4c = f(c)$$

So the limit can be found easily by evaluating the function f at c.

This simple method for finding limits is very useful, because there are many functions that satisfy the property

$$\lim_{x \to c} f(x) = f(c) \tag{1}$$

Any polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

satisfies (1) for any real number c. Also, any rational function

$$r(x) = \frac{n(x)}{d(x)}$$

where $n(x)$ and $d(x)$ are polynomials, satisfies (1) provided c is a real number for which $d(c) \neq 0$.

Limits: An Algebraic Approach

THEOREM 3 Limits of Polynomial and Rational Functions

1. $\lim_{x \to c} f(x) = f(c)$ for f any polynomial function.
2. $\lim_{x \to c} r(x) = r(c)$ for r any rational function with a nonzero denominator at $x = c$.

EXAMPLE 6 Evaluating Limits Find each limit.

(A) $\lim_{x \to 2} (x^3 - 5x - 1)$ (B) $\lim_{x \to -1} \sqrt{2x^2 + 3}$ (C) $\lim_{x \to 4} \frac{2x}{3x + 1}$

SOLUTION

(A) $\lim_{x \to 2} (x^3 - 5x - 1) = 2^3 - 5 \cdot 2 - 1 = -3$ Theorem 3

(B)
$$\lim_{x \to -1} \sqrt{2x^2 + 3} = \sqrt{\lim_{x \to -1} (2x^2 + 3)}$$ Property 8
$$= \sqrt{2(-1)^2 + 3}$$ Theorem 3
$$= \sqrt{5}$$

(C)
$$\lim_{x \to 4} \frac{2x}{3x + 1} = \frac{2 \cdot 4}{3 \cdot 4 + 1}$$ Theorem 3
$$= \frac{8}{13}$$

Limits: An Algebraic Approach

EXAMPLE 7 Evaluating Limits Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 2 \\ x - 1 & \text{if } x > 2 \end{cases}$$

Find:

(A) $\lim_{x \to 2^-} f(x)$ (B) $\lim_{x \to 2^+} f(x)$ (C) $\lim_{x \to 2} f(x)$ (D) $f(2)$

SOLUTION

(A) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 + 1)$ If $x < 2$, $f(x) = x^2 + 1$.

$= 2^2 + 1 = 5$

(B) $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x - 1)$ If $x > 2$, $f(x) = x - 1$.

$= 2 - 1 = 1$

(C) Since the one-sided limits are not equal, $\lim_{x \to 2} f(x)$ does not exist.

(D) Because the definition of f does not assign a value to f for $x = 2$, only for $x < 2$ and $x > 2$, $f(2)$ does not exist.

Limits: An Algebraic Approach

Matched Problem 7 Let

$$f(x) = \begin{cases} 2x + 3 & \text{if } x < 5 \\ -x + 12 & \text{if } x > 5 \end{cases}$$

Find:

(A) $\lim_{x \to 5^-} f(x)$ (B) $\lim_{x \to 5^+} f(x)$ (C) $\lim_{x \to 5} f(x)$ (D) $f(5)$

EXAMPLE

Let $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 2x & \text{if } x > 1 \end{cases}$. Find

(A) $\lim_{x \to 1^+} f(x)$ (B) $\lim_{x \to 1^-} f(x)$

(C) $\lim_{x \to 1} f(x)$ (D) $f(1)$

It is important to note that there are restrictions on some of the limit properties. In particular, if

$$\lim_{x \to c} f(x) = 0 \quad \text{and} \quad \lim_{x \to c} g(x) = 0, \quad \text{then finding } \lim_{x \to c} \frac{f(x)}{g(x)}$$

may present some difficulties, since limit property 7 (the limit of a quotient) does not apply when $\lim_{x \to c} g(x) = 0$. The next example illustrates some techniques that can be useful in this situation.

Limits: An Algebraic Approach

EXAMPLE 8 Evaluating Limits Find each limit.

(A) $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$ (B) $\lim_{x \to -1} \frac{x|x + 1|}{x + 1}$

SOLUTION

(A) Note that $\lim_{x \to 2} x^2 - 4 = 2^2 - 4 = 0$ and $\lim_{x \to 2} x - 2 = 2 - 2 = 0$. Algebraic simplification is often useful in such a case when the numerator and denominator both have limit 0.

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

(B) One-sided limits are helpful for limits involving the absolute value function.

$$\lim_{x \to -1^+} \frac{x|x + 1|}{x + 1} = \lim_{x \to -1^+} (x) = -1 \quad \text{If } x > -1, \text{ then } \frac{|x + 1|}{x + 1} = 1.$$

$$\lim_{x \to -1^-} \frac{x|x + 1|}{x + 1} = \lim_{x \to -1^-} (-x) = 1 \quad \text{If } x < -1, \text{ then } \frac{|x + 1|}{x + 1} = -1.$$

Since the limit from the left and the limit from the right are not the same, we conclude that

$$\lim_{x \to -1} \frac{x|x + 1|}{x + 1} \quad \text{does not exist}$$

Limits: An Algebraic Approach

Matched Problem 8 Find each limit.

(A) $\lim_{x \to -3} \frac{x^2 + 4x + 3}{x + 3}$ (B) $\lim_{x \to 4} \frac{x^2 - 16}{|x - 4|}$

In the solution to Example 8A we used the following algebraic identity:

$$\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2, \quad x \neq 2$$

The restriction $x \neq 2$ is necessary here because the first two expressions are not defined at $x = 2$. Why didn't we include this restriction in the solution? When x approaches 2 in a limit problem, it is assumed that x is close, but not equal, to 2. It is important that you understand that both of the following statements are valid:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) \quad \text{and} \quad \frac{x^2 - 4}{x - 2} = x + 2, \quad x \neq 2$$

Limits like those in Example 8 occur so frequently in calculus that they are given a special name.

DEFINITION Indeterminate Form

If $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$, then $\lim_{x \to c} \frac{f(x)}{g(x)}$ is said to be **indeterminate**, or, more specifically, a **0/0 indeterminate form**.

Limits: An Algebraic Approach

The term *indeterminate* is used because the limit of an indeterminate form may or may not exist (see Example 8A and 8B).

CAUTION The expression $0/0$ does not represent a real number and should never be used as the value of a limit. If a limit is a $0/0$ indeterminate form, further investigation is always required to determine whether the limit exists and to find its value if it does exist.

If the denominator of a quotient approaches 0 and the numerator approaches a nonzero number, then the limit of the quotient is not an indeterminate form. In fact, in this case the limit of the quotient does not exist.

THEOREM 4 Limit of a Quotient

If $\lim_{x \to c} f(x) = L, L \neq 0$, and $\lim_{x \to c} g(x) = 0$,

then

$$\lim_{x \to c} \frac{f(x)}{g(x)} \qquad \text{does not exist}$$

Limits: An Algebraic Approach

EXAMPLE 9 Indeterminate Forms Is the limit expression a 0/0 indeterminate form? Find the limit or explain why the limit does not exist.

(A) $\lim_{x \to 1} \frac{x-1}{x^2+1}$ (B) $\lim_{x \to 1} \frac{x-1}{x^2-1}$ (C) $\lim_{x \to 1} \frac{x+1}{x^2-1}$

SOLUTION

(A) $\lim_{x \to 1} (x-1) = 0$ but $\lim_{x \to 1} (x^2+1) = 2$. So no, the limit expression is not a 0/0 indeterminate form. By property 7 of Theorem 2,

$$\lim_{x \to 1} \frac{x-1}{x^2+1} = \frac{0}{2} = 0$$

(B) $\lim_{x \to 1} (x-1) = 0$ and $\lim_{x \to 1} (x^2-1) = 0$. So yes, the limit expression is a 0/0 indeterminate form. We factor x^2-1 to simplify the limit expression and find the limit:

$$\lim_{x \to 1} \frac{x-1}{x^2-1} = \lim_{x \to 1} \frac{x-1}{(x-1)(x+1)} = \lim_{x \to 1} \frac{1}{x+1} = \frac{1}{2}$$

Limits: An Algebraic Approach

(C) $\lim_{x \to 1} (x + 1) = 2$ and $\lim_{x \to 1} (x^2 - 1) = 0$. So no, the limit expression is not a $0/0$ indeterminate form. By Theorem 4,

$$\lim_{x \to 1} \frac{x + 1}{x^2 - 1} \quad \text{does not exist}$$

Matched Problem 9 Is the limit expression a $0/0$ indeterminate form? Find the limit or explain why the limit does not exist.

(A) $\lim_{x \to 3} \frac{x + 1}{x + 3}$ (B) $\lim_{x \to 3} \frac{x - 3}{x^2 + 9}$ (C) $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

Limits of Difference Quotients

Let the function f be defined in an open interval containing the number a. One of the most important limits in calculus is the limit of the **difference quotient**,

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{2}$$

If

$$\lim_{h \to 0} [f(a + h) - f(a)] = 0$$

as it often does, then limit (2) is an indeterminate form.

Limits: An Algebraic Approach

EXAMPLE 10 Limit of a Difference Quotient Find the following limit for $f(x) = 4x - 5$:

$$\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

SOLUTION

$$\lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{[4(\mathbf{3}+\boldsymbol{h}) - 5] - [4(\mathbf{3}) - 5]}{h}$$

$$= \lim_{h \to 0} \frac{12 + 4h - 5 - 12 + 5}{h}$$

$$= \lim_{h \to 0} \frac{4h}{h} = \lim_{h \to 0} 4 = 4$$

Since this is a 0/0 indeterminate form and property 7 in Theorem 2 does not apply, we proceed with algebraic simplification.

Matched Problem 10 Find the following limit for $f(x) = 7 - 2x$:

$$\lim_{h \to 0} \frac{f(4+h) - f(4)}{h}.$$

Explore and Discuss 2

If $f(x) = \frac{1}{x}$, explain why $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = -\frac{1}{9}$.

In Problems 9–16, use the graph of the function f shown to estimate the indicated limits and function values.

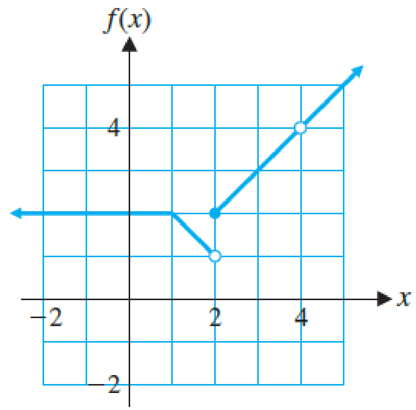


Figure for 9–16

9. $f(-0.5)$

10. $f(-1.5)$

11. $f(1.75)$

12. $f(1.25)$

13. (A) $\lim_{x \to 0^-} f(x)$ (B) $\lim_{x \to 0^+} f(x)$

(C) $\lim_{x \to 0} f(x)$ (D) $f(0)$

14. (A) $\lim_{x \to 1^-} f(x)$ (B) $\lim_{x \to 1^+} f(x)$

(C) $\lim_{x \to 1} f(x)$ (D) $f(1)$

15. (A) $\lim_{x \to 2^-} f(x)$ (B) $\lim_{x \to 2^+} f(x)$

(C) $\lim_{x \to 2} f(x)$ (D) $f(2)$

(E) Is it possible to redefine $f(2)$ so that $\lim_{x \to 2} f(x) = f(2)$? Explain.

16. (A) $\lim_{x \to 4^-} f(x)$ (B) $\lim_{x \to 4^+} f(x)$

(C) $\lim_{x \to 4} f(x)$ (D) $f(4)$

(E) Is it possible to define $f(4)$ so that $\lim_{x \to 4} f(x) = f(4)$? Explain.

Problem

Telephone rates. A long-distance telephone service charges \$0.99 for the first 20 minutes or less of a call and \$0.07 per minute thereafter.

(A) Write a piecewise definition of the charge $F(x)$ for a long-distance call lasting x minutes.

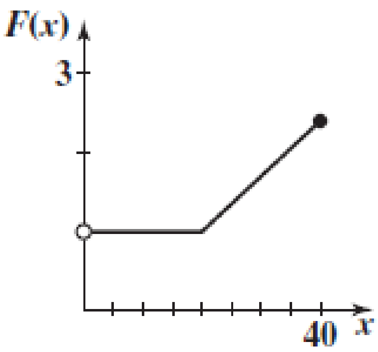
(B) Graph $F(x)$ for $0 < x \leq 40$.

(C) Find $\lim_{x \to 20^-} F(x)$, $\lim_{x \to 20^+} F(x)$, and $\lim_{x \to 20} F(x)$, whichever exist.

Answer

(A) $F(x) = \begin{cases} 0.99 & \text{if } 0 < x \leq 20 \\ 0.07x - 0.41 & \text{if } x \geq 20 \end{cases}$

(B)



(C) All 3 limits are 0.99.

Problem

Pollution A state charges polluters an annual fee of \$20 per ton for each ton of pollutant emitted into the atmosphere, up to a maximum of 4,000 tons. No fees are charged for emissions beyond the 4,000-ton limit. Write a piecewise definition of the fees $F(x)$ charged for the emission of x tons of pollutant in a year. What is the limit of $F(x)$ as x approaches 4,000 tons? As x approaches 8,000 tons?

Answer

$$F(x) = \begin{cases} 20x & \text{if } 0 < x \le 4{,}000 \\ 80{,}000 & \text{if } x \ge 4{,}000 \end{cases}$$

$$\lim_{x \to 4{,}000} F(x) = 80{,}000; \quad \lim_{x \to 8{,}000} F(x) = 80{,}000$$