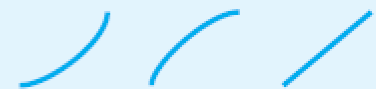
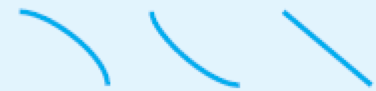
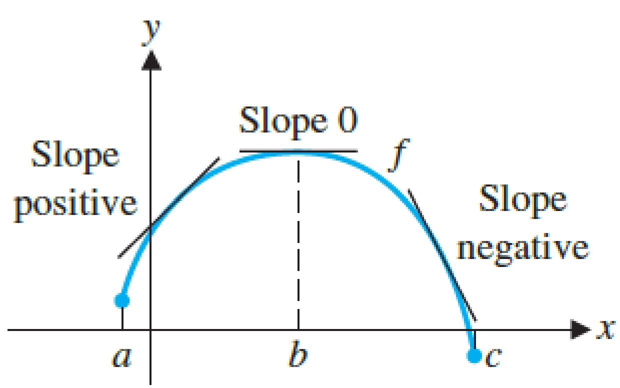
First Derivative and Graphs

Increasing and Decreasing Functions

THEOREM 1 Increasing and Decreasing Functions

For the interval (a, b), if $f' > 0$, then f is increasing, and if $f' < 0$, then f is decreasing.

$f'(x)$	$f(x)$	Graph of f	Examples
+	Increases ↗	Rises ↗	
−	Decreases ↘	Falls ↘	



Increasing and Decreasing Functions

EXAMPLE 1 Finding Intervals on Which a Function Is Increasing or Decreasing

Given the function $f(x) = 8x - x^2$,

(A) Which values of x correspond to horizontal tangent lines?

(B) For which values of x is $f(x)$ increasing? Decreasing?

(C) Sketch a graph of f. Add any horizontal tangent lines.

SOLUTION

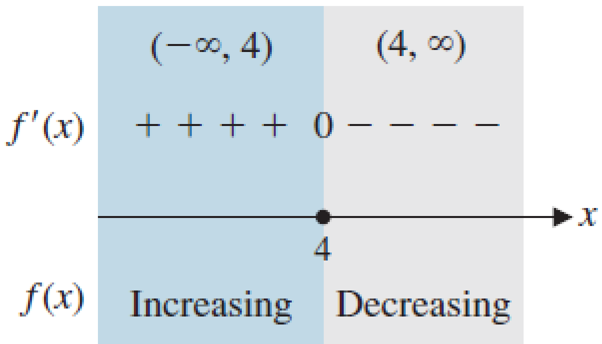
(A) $$f'(x) = 8 - 2x = 0$$
$$x = 4$$

So, a horizontal tangent line exists at $x = 4$ only.

(B) We will construct a sign chart for $f'(x)$ to determine which values of x make $f'(x) > 0$ and which values make $f'(x) < 0$. Recall from Section 2.3 that the partition numbers for a function are the numbers at which the function is 0 or discontinuous. When constructing a sign chart for $f'(x)$, we must locate all points where $f'(x) = 0$ or $f'(x)$ is discontinuous. From part (A), we know that $f'(x) = 8 - 2x = 0$ at $x = 4$. Since $f'(x) = 8 - 2x$ is a polynomial, it is continuous for all x. So, 4 is the only partition number for f'. We construct a sign chart for the intervals $(-\infty, 4)$ and $(4, \infty)$, using test numbers 3 and 5:

Increasing and Decreasing Functions

SOLUTION

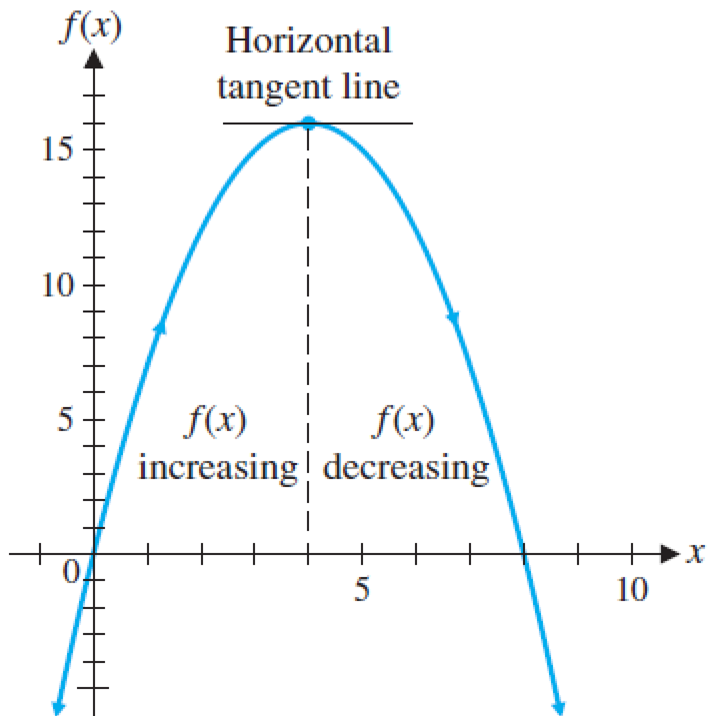


Test Numbers	
x	$f'(x)$
3	2 (+)
5	−2 (−)

Therefore, $f(x)$ is increasing on $(-\infty, 4)$ and decreasing on $(4, \infty)$.

(C)

x	$f(x)$
0	0
2	12
4	16
6	12
8	0



Increasing and Decreasing Functions

DEFINITION Critical Numbers

A real number x in the domain of f such that $f'(x) = 0$ or $f'(x)$ does not exist is called a **critical number** of f.

CONCEPTUAL INSIGHT

The critical numbers of f belong to the domain of f and are partition numbers for f'. But f' may have partition numbers that do not belong to the domain of f, so are not critical numbers of f.

If f is a polynomial, then both the partition numbers for f' and the critical numbers of f are the solutions of $f'(x) = 0$.

Increasing and Decreasing Functions

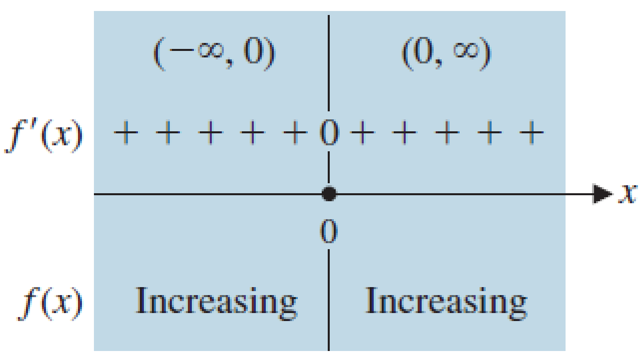
EXAMPLE 2 Partition Numbers for f' and Critical Numbers of f Find the critical numbers of f, the intervals on which f is increasing, and those on which f is decreasing, for $f(x) = 1 + x^3$.

SOLUTION Begin by finding the partition numbers for $f'(x)$ [since $f'(x) = 3x^2$ is continuous we just need to solve $f'(x) = 0$]

$$f'(x) = 3x^2 = 0 \quad \text{only if } x = 0$$

The partition number 0 for f' is in the domain of f, so 0 is the only critical number of f.

The sign chart for $f'(x) = 3x^2$ (partition number is 0) is



Test Numbers	
x	$f'(x)$
-1	$3 \quad (+)$
1	$3 \quad (+)$

The sign chart indicates that $f(x)$ is increasing on $(-\infty, 0)$ and $(0, \infty)$. Since f is continuous at $x = 0$, it follows that $f(x)$ is increasing for all x. The graph of f is shown in Figure 3.

Increasing and Decreasing Functions

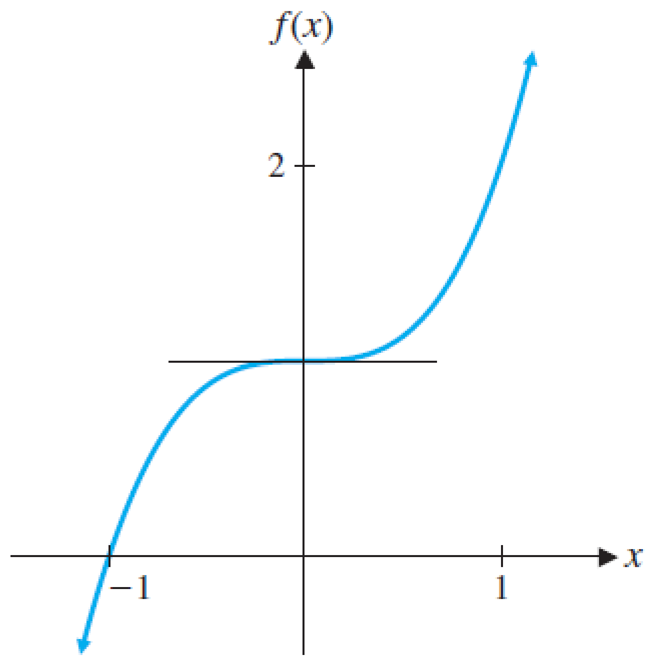


Figure 3

Matched Problem 2 Find the critical numbers of f, the intervals on which f is increasing, and those on which f is decreasing, for $f(x) = 1 - x^3$.

Increasing and Decreasing Functions

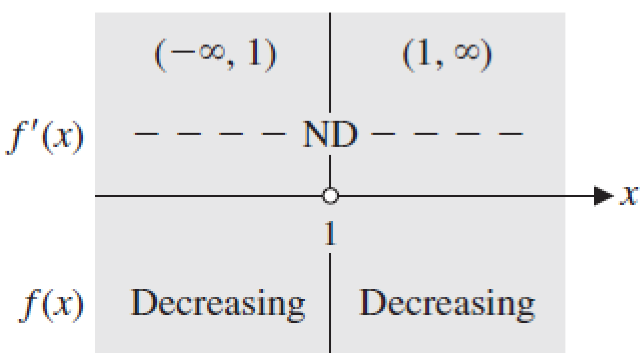
EXAMPLE 3 **Partition Numbers for f' and Critical Numbers of f** Find the critical numbers of f, the intervals on which f is increasing, and those on which f is decreasing, for $f(x) = (1 - x)^{1/3}$.

SOLUTION

$$f'(x) = -\frac{1}{3}(1 - x)^{-2/3} = \frac{-1}{3(1 - x)^{2/3}}$$

To find the partition numbers for f', we note that f' is continuous for all x, except for values of x for which the denominator is 0; that is, $f'(1)$ does not exist and f' is discontinuous at $x = 1$. Since the numerator of f' is the constant -1, $f'(x) \neq 0$ for any value of x. Thus, $x = 1$ is the only partition number for f'. Since 1 is in the domain of f, $x = 1$ is also the only critical number of f. When constructing the sign chart for f' we use the abbreviation ND to note the fact that $f'(x)$ is *not defined* at $x = 1$.

The sign chart for $f'(x) = -1/[3(1 - x)^{2/3}]$ (partition number for f' is 1) is as follows:



Test Numbers

x	$f'(x)$
0	$-\frac{1}{3}$ (−)
2	$-\frac{1}{3}$ (−)

Increasing and Decreasing Functions

SOLUTION

The sign chart indicates that f is decreasing on $(-\infty, 1)$ and $(1, \infty)$. Since f is continuous at $x = 1$, it follows that $f(x)$ is decreasing for all x. **A continuous function can be decreasing (or increasing) on an interval containing values of x where $f'(x)$ does not exist.** The graph of f is shown in Figure 4. Notice that the undefined derivative at $x = 1$ results in a vertical tangent line at $x = 1$. **A vertical tangent will occur at $x = c$ if f is continuous at $x = c$ and if $|f'(x)|$ becomes larger and larger as x approac**

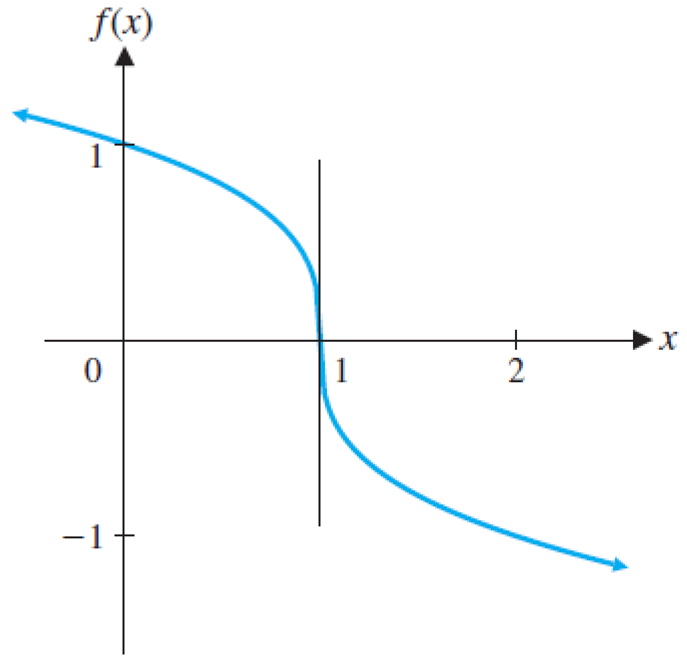
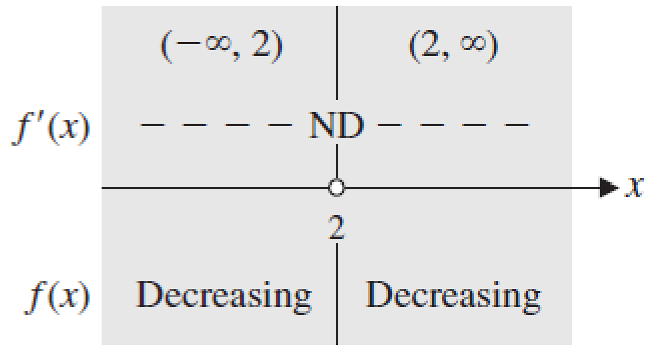


Figure 4

Increasing and Decreasing Functions

SOLUTION

The sign chart for $f'(x) = -1/(x-2)^2$ (partition number for f' is 2) is as follows:



Test Numbers	
x	$f'(x)$
1	-1 $(-)$
3	-1 $(-)$

Therefore, f is decreasing on $(-\infty, 2)$ and $(2, \infty)$. The graph of f is shown in Figure 5.

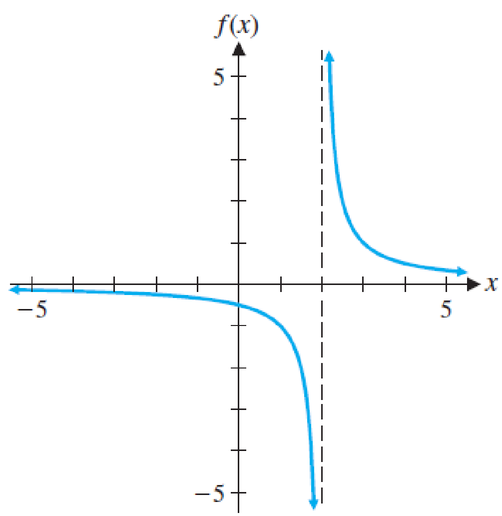


Figure 5

Increasing and Decreasing Functions

Matched Problem 3 Find the critical numbers of f, the intervals on which f is increasing, and those on which f is decreasing, for $f(x) = (1 + x)^{1/3}$.

EXAMPLE 4 Partition Numbers for f' and Critical Numbers of f Find the critical numbers of f, the intervals on which f is increasing, and those on which f is decreasing, for $f(x) = \dfrac{1}{x - 2}$.

SOLUTION

$$f(x) = \frac{1}{x - 2} = (x - 2)^{-1}$$

$$f'(x) = -(x - 2)^{-2} = \frac{-1}{(x - 2)^2}$$

To find the partition numbers for f', note that $f'(x) \neq 0$ for any x and f' is not defined at $x = 2$. Thus, $x = 2$ is the only partition number for f'. However, $x = 2$ is *not* in the domain of f. Consequently, $x = 2$ is *not* a critical number of f. The function f has no critical numbers.

Increasing and Decreasing Functions

EXAMPLE 5 Partition Numbers for f' and Critical Numbers of f Find the critical numbers of f, the intervals on which f is increasing, and those on which f is decreasing, for $f(x) = 8 \ln x - x^2$.

SOLUTION The natural logarithm function $\ln x$ is defined on $(0, \infty)$, or $x > 0$, so $f(x)$ is defined only for $x > 0$.

$$f(x) = 8 \ln x - x^2, x > 0$$

$$f'(x) = \frac{8}{x} - 2x \qquad \text{Find a common denominator.}$$

$$= \frac{8}{x} - \frac{2x^2}{x} \qquad \text{Subtract numerators.}$$

$$= \frac{8 - 2x^2}{x} \qquad \text{Factor numerator.}$$

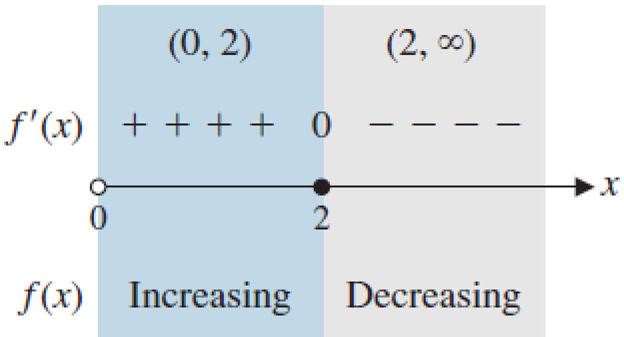
$$= \frac{2(2 - x)(2 + x)}{x}, \quad x > 0$$

The only partition number for f' that is positive, and therefore belongs to the domain of f, is 2. So 2 is the only critical number of f.

Increasing and Decreasing Functions

SOLUTION

The sign chart for $f'(x) = \dfrac{2(2-x)(2+x)}{x}$, $x > 0$ (partition number for f' is 2), is as follows:



Test Numbers	
x	$f'(x)$
1	6 (+)
4	−6 (−)

Therefore, f is increasing on $(0, 2)$ and decreasing on $(2, \infty)$.

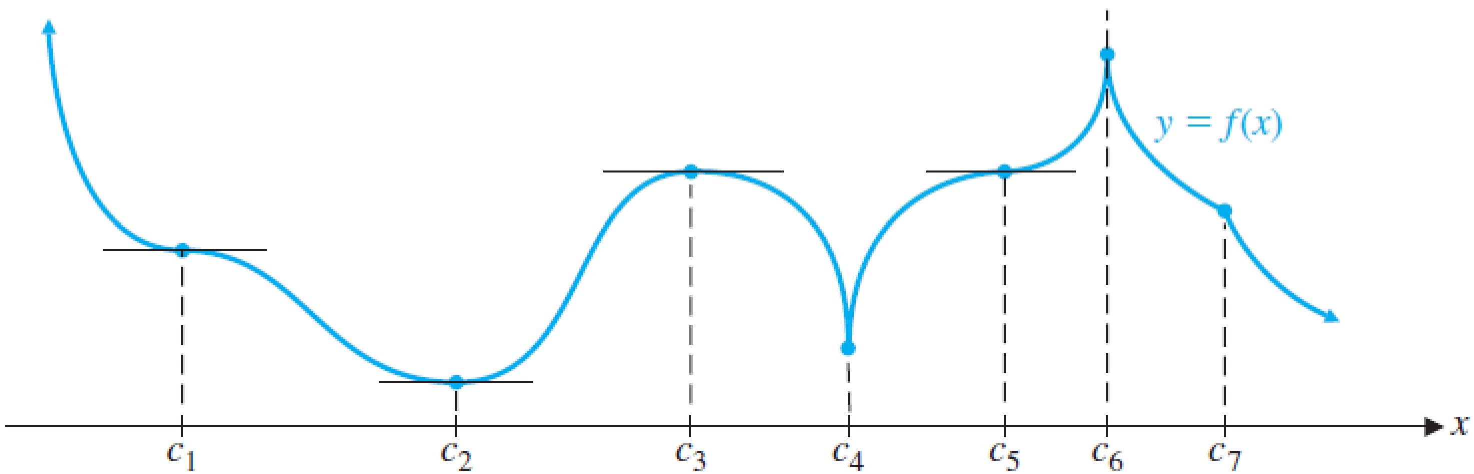
Matched Problem 5 Find the critical numbers of f, the intervals on which f is increasing, and those on which f is decreasing, for $f(x) = 5 \ln x - x$.

Examples 4 and 5 illustrate two important ideas:

1. Do not assume that all partition numbers for the derivative f' are critical numbers of the function f. To be a critical number of f, a partition number for f' must also be in the domain of f.
2. The intervals on which a function f is increasing or decreasing must always be expressed in terms of open intervals that are subsets of the domain of f.

Local Extrema

Local Extrema

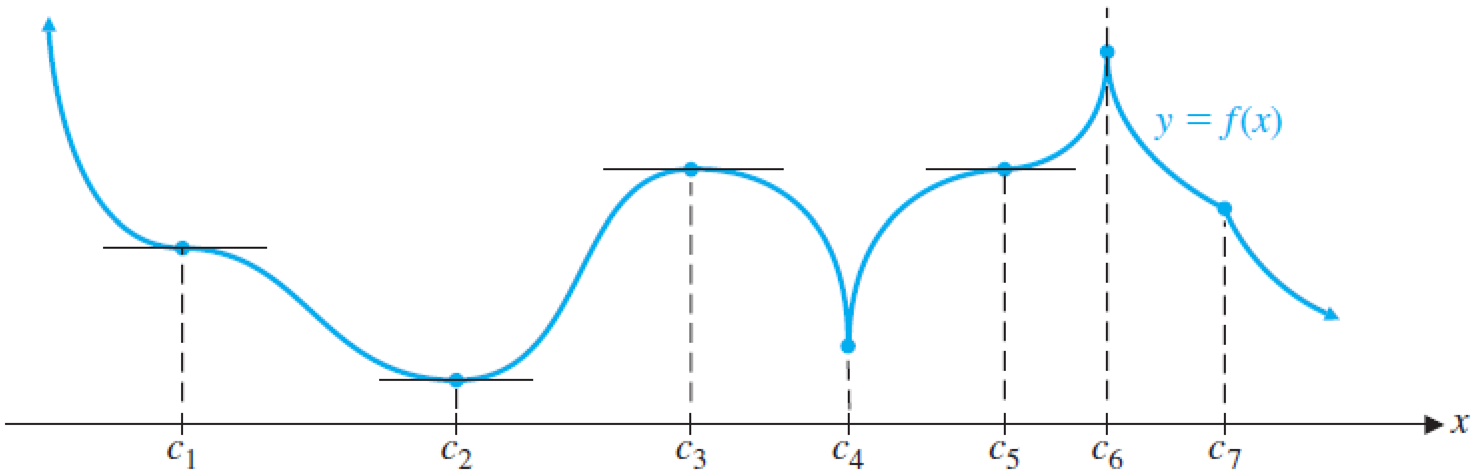


When the graph of a continuous function changes from rising to falling, a high point, or *local maximum*, occurs. When the graph changes from falling to rising, a low point, or *local minimum*, occurs. In Figure 7, high points occur at c_3 and c_6, and low points occur at c_2 and c_4. In general, we call $f(c)$ a **local maximum** if there exists an interval (m, n) containing c such that

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } (m, n)$$

Note that this inequality need hold only for numbers x near c, which is why we use the term *local*. So the y coordinate of the high point $(c_3, f(c_3))$ in Figure 7 is a local maximum, as is the y coordinate of $(c_6, f(c_6))$.

Local Extrema



The value $f(c)$ is called a **local minimum** if there exists an interval (m, n) containing c such that

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } (m, n)$$

The value $f(c)$ is called a **local extremum** if it is either a local maximum or a local minimum. A point on a graph where a local extremum occurs is also called a **turning point**. In Figure 7 we see that local maxima occur at c_3 and c_6, local minima occur at c_2 and c_4, and all four values produce local extrema. Also, the local maximum $f(c_3)$ is not the largest y coordinate of points on the graph in Figure 7. Later in this chapter, we consider the problem of finding *absolute extrema*, the y coordinates of the highest and lowest points on a graph. For now, we are concerned only with locating *local* extrema.

Local Extrema

EXAMPLE 6 Analyzing a Graph Use the graph of f in Figure 8 to find the intervals on which f is increasing, those on which f is decreasing, any local maxima, and any local minima.

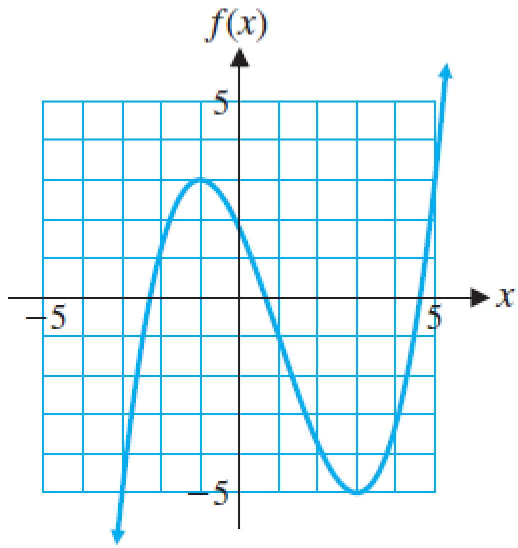


Figure 8

SOLUTION The function f is increasing (the graph is rising) on $(-\infty, -1)$ and on $(3, \infty)$ and is decreasing (the graph is falling) on $(-1, 3)$. Because the graph changes from rising to falling at $x = -1$, $f(-1) = 3$ is a local maximum. Because the graph changes from falling to rising at $x = 3$, $f(3) = -5$ is a local minimum.

Local Extrema

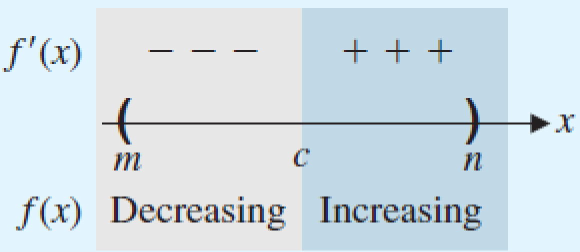
THEOREM 2 Local Extrema and Critical Numbers

If $f(c)$ is a local extremum of the function f, then c is a critical number of f.

PROCEDURE First-Derivative Test for Local Extrema

Let c be a critical number of f [$f(c)$ is defined and either $f'(c) = 0$ or $f'(c)$ is not defined]. Construct a sign chart for $f'(x)$ close to and on either side of c.

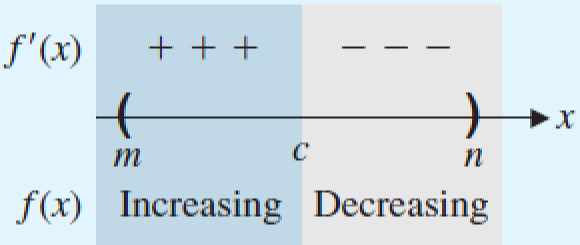
Sign Chart



$f(c)$

$f(c)$ is a local minimum.

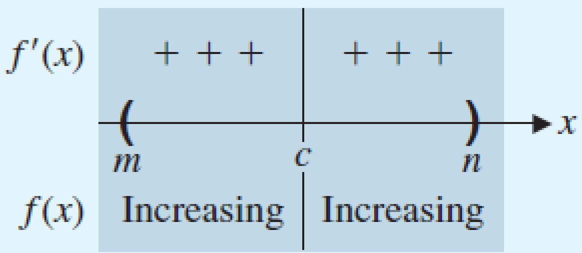
If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a local minimum.



$f(c)$ is a local maximum.

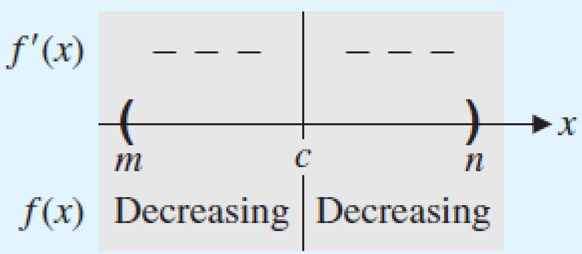
If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a local maximum.

Local Extrema



$f(c)$ is not a local extremum.

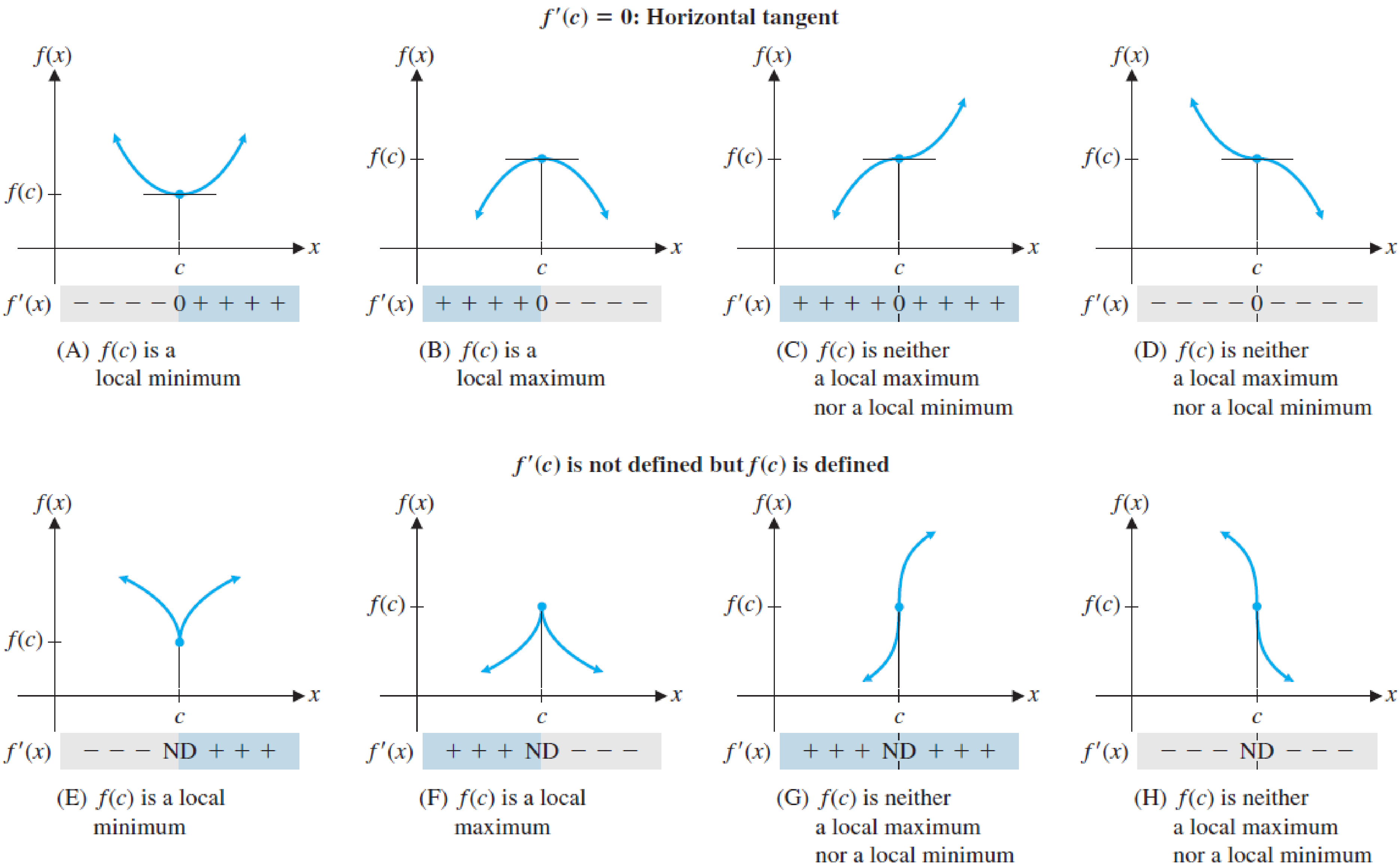
If $f'(x)$ does not change sign at c, then $f(c)$ is neither a local maximum nor a local minimum.



$f(c)$ is not a local extremum.

If $f'(x)$ does not change sign at c, then $f(c)$ is neither a local maximum nor a local minimum.

Local Extrema



Local Extrema

EXAMPLE 7 Locating Local Extrema Given $f(x) = x^3 - 6x^2 + 9x + 1$,

(A) Find the critical numbers of f.

(B) Find the local maxima and local minima of f.

SOLUTION

(A) Find all numbers x in the domain of f where $f'(x) = 0$ or $f'(x)$ does not exist.

$$f'(x) = 3x^2 - 12x + 9 = 0$$
$$3(x^2 - 4x + 3) = 0$$
$$3(x - 1)(x - 3) = 0$$
$$x = 1 \quad \text{or} \quad x = 3$$

$f'(x)$ exists for all x; the critical numbers of f are $x = 1$ and $x = 3$.

Local Extrema

EXAMPLE 7 Locating Local Extrema Given $f(x) = x^3 - 6x^2 + 9x + 1$,

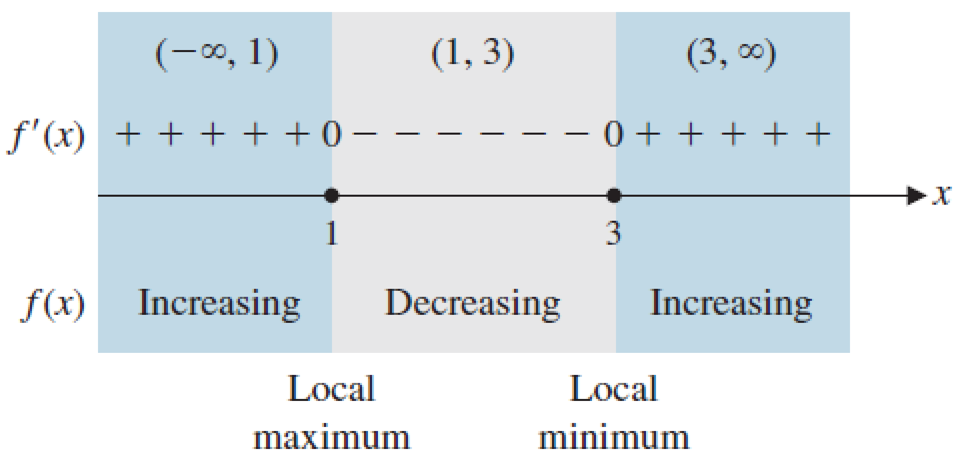
(A) Find the critical numbers of f.

(B) Find the local maxima and local minima of f.

SOLUTION

(B) The easiest way to apply the first-derivative test for local maxima and minima is to construct a sign chart for $f'(x)$ for all x. Partition numbers for $f'(x)$ are $x = 1$ and $x = 3$ (which also happen to be critical numbers of f).

Sign chart for $f'(x) = 3(x - 1)(x - 3)$:



Test Numbers

x	$f'(x)$
0	9 (+)
2	−3 (−)
4	9 (+)

Local Extrema

SOLUTION

The sign chart indicates that f increases on $(-\infty, 1)$, has a local maximum at $x = 1$, decreases on $(1, 3)$, has a local minimum at $x = 3$, and increases on $(3, \infty)$. These facts are summarized in the following table:

x	$f'(x)$	$f(x)$	Graph of f
$(-\infty, 1)$	$+$	Increasing	Rising
$x = 1$	0	Local maximum	Horizontal tangent
$(1, 3)$	$-$	Decreasing	Falling
$x = 3$	0	Local minimum	Horizontal tangent
$(3, \infty)$	$+$	Increasing	Rising

The local maximum is $f(1) = 5$; the local minimum is $f(3) = 1$.

Matched Problem 7 Given $f(x) = x^3 - 9x^2 + 24x - 10$,

(A) Find the critical numbers of f.

(B) Find the local maxima and local minima of f.

(C) Sketch a graph of f.

Local Extrema

EXAMPLE 9 Revenue Analysis The graph of the total revenue $R(x)$ (in dollars) from the sale of x bookcases is shown in Figure 14.

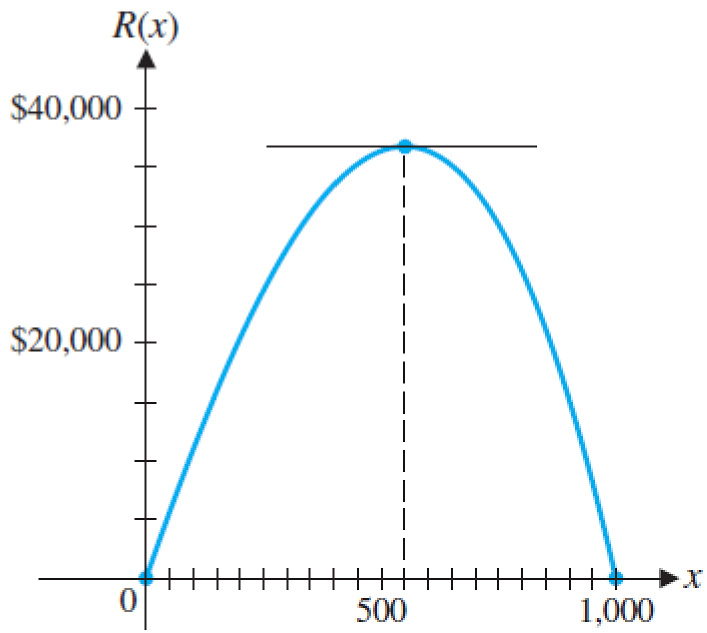


Figure 14 **Revenue**

(A) Write a brief description of the graph of the marginal revenue function $y = R'(x)$, including a discussion of any x intercepts.

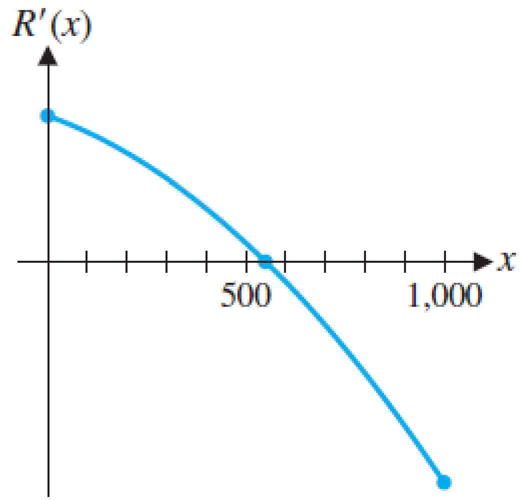
(B) Sketch a possible graph of $y = R'(x)$.

Local Extrema

SOLUTION

(A) The graph of $y = R(x)$ indicates that $R(x)$ increases on $(0, 550)$, has a local maximum at $x = 550$, and decreases on $(550, 1{,}000)$. Consequently, the marginal revenue function $R'(x)$ must be positive on $(0, 550)$, 0 at $x = 550$, and negative on $(550, 1{,}000)$.

(B) A possible graph of $y = R'(x)$ illustrating the information summarized in part (A) is shown in Figure 15.



Second Derivative and Graphs

Concavity

DEFINITION Concavity

The graph of a function f is **concave upward** on the interval (a, b) if $f'(x)$ is *increasing* on (a, b) and is **concave downward** on the interval (a, b) if $f'(x)$ is *decreasing* on (a, b).

Using Concavity as a Graphing Tool

Consider the functions

$$f(x) = x^2 \qquad \text{and} \qquad g(x) = \sqrt{x}$$

for x in the interval $(0, \infty)$. Since

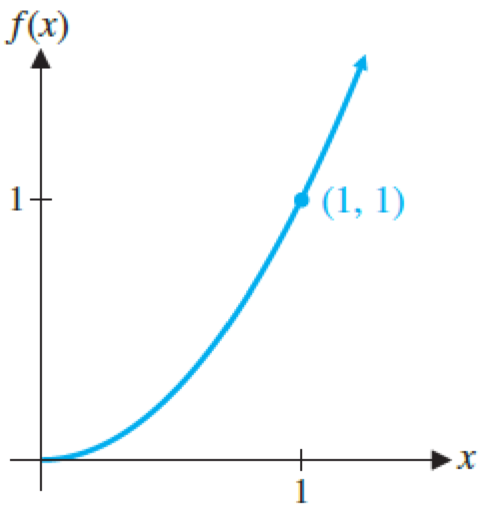
$$f'(x) = 2x > 0 \qquad \text{for } 0 < x < \infty$$

and

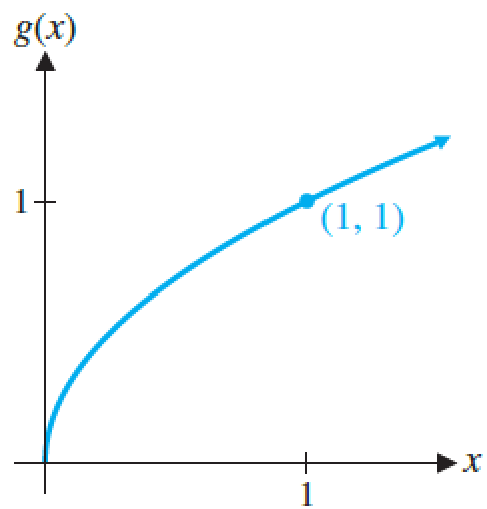
$$g'(x) = \frac{1}{2\sqrt{x}} > 0 \qquad \text{for } 0 < x < \infty$$

both functions are increasing on $(0, \infty)$.

Concavity



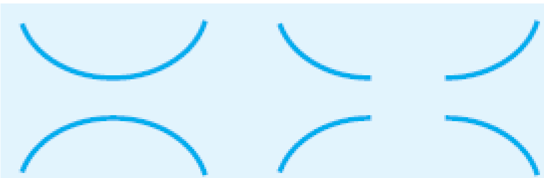
(A) $f(x) = x^2$



(B) $g(x) = \sqrt{x}$

SUMMARY Concavity

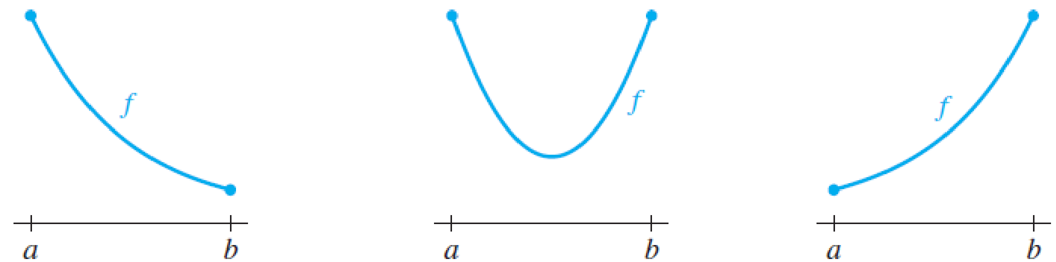
For the interval (a, b), if $f'' > 0$, then f is concave upward, and if $f'' < 0$, then f is concave downward.

$f''(x)$	$f'(x)$	Graph of $y = f(x)$	Examples
+	Increasing	Concave upward	
−	Decreasing	Concave downward	

CONCEPTUAL INSIGHT

Be careful not to confuse concavity with falling and rising. A graph that is concave upward on an interval may be falling, rising, or both falling and rising on that interval. A similar statement holds for a graph that is concave downward. See Figure 3.

$f''(x) > 0$ on (a, b)
Concave upward

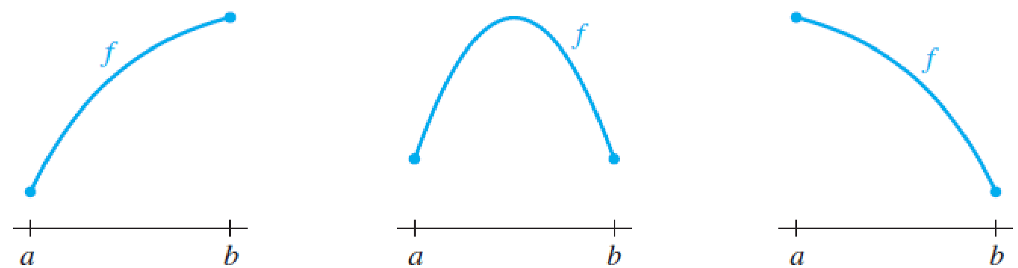


(A) $f'(x)$ is negative and increasing. Graph of f is falling.

(B) $f'(x)$ increases from negative to positive. Graph of f falls, then rises.

(C) $f'(x)$ is positive and increasing. Graph of f is rising.

$f''(x) < 0$ on (a, b)
Concave downward



(D) $f'(x)$ is positive and decreasing. Graph of f is rising.

(E) $f'(x)$ decreases from positive to negative. Graph of f rises, then falls.

(F) $f'(x)$ is negative and decreasing. Graph of f is falling.

Figure 3 **Concavity**

Concavity

EXAMPLE 1 Concavity of Graphs Determine the intervals on which the graph of each function is concave upward and the intervals on which it is concave downward. Sketch a graph of each function.

(A) $f(x) = e^x$

(B) $g(x) = \ln x$

(C) $h(x) = x^3$

SOLUTION

(A) $f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

Since $f''(x) > 0$ on $(-\infty, \infty)$, the graph of $f(x) = e^x$ [Fig. 4(A)] is concave upward on $(-\infty, \infty)$.

(B) $g(x) = \ln x$

$g'(x) = \frac{1}{x}$

$g''(x) = -\frac{1}{x^2}$

The domain of $g(x) = \ln x$ is $(0, \infty)$ and $g''(x) < 0$ on this interval, so the graph of $g(x) = \ln x$ [Fig. 4(B)] is concave downward on $(0, \infty)$.

(C) $h(x) = x^3$

$h'(x) = 3x^2$

$h''(x) = 6x$

Since $h''(x) < 0$ when $x < 0$ and $h''(x) > 0$ when $x > 0$, the graph of $h(x) = x^3$ [Fig. 4(C)] is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$.

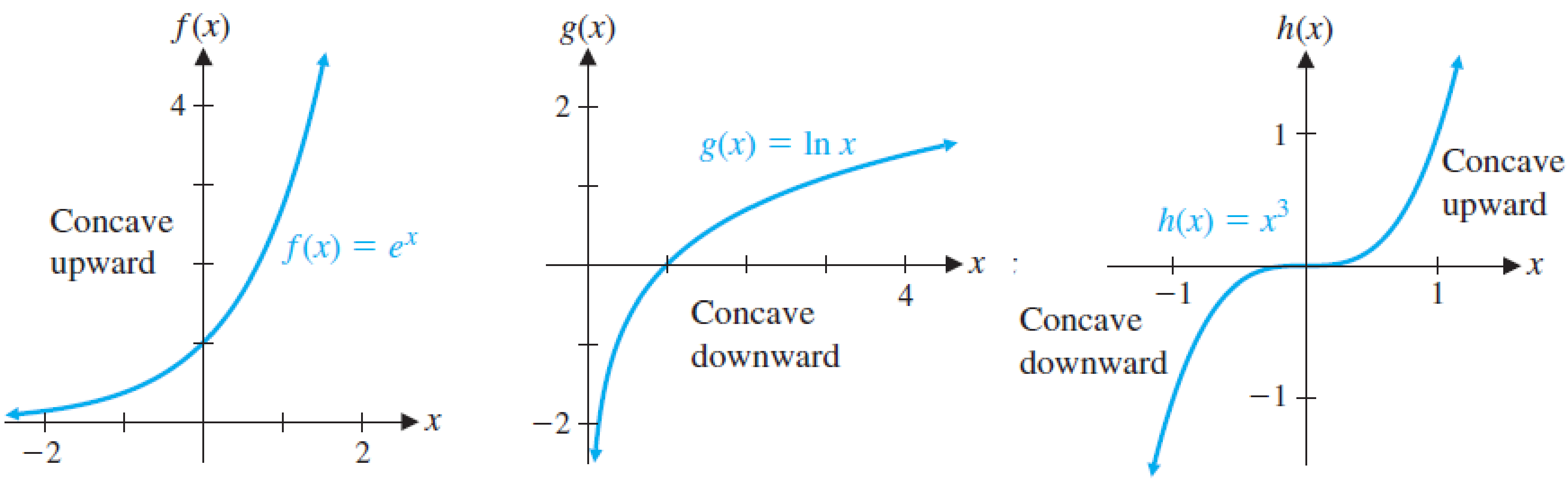
Concavity

EXAMPLE 1 Concavity of Graphs Determine the intervals on which the graph of each function is concave upward and the intervals on which it is concave downward. Sketch a graph of each function.

(A) $f(x) = e^x$

(B) $g(x) = \ln x$

(C) $h(x) = x^3$



(A) Concave upward for all x.

(B) Concave downward for $x > 0$.

(C) Concavity changes at the origin.

Inflection Points

An **inflection point** is a point on the graph of a function where the concavity changes (from upward to downward or from downward to upward). For the concavity to change at a point, $f''(x)$ must change sign at that point.

THEOREM 1 Inflection Points

If $(c, f(c))$ is an inflection point of f, then c is a partition number for f''.

Our strategy for finding inflection points is clear: We find all partition numbers c for f'' and ask

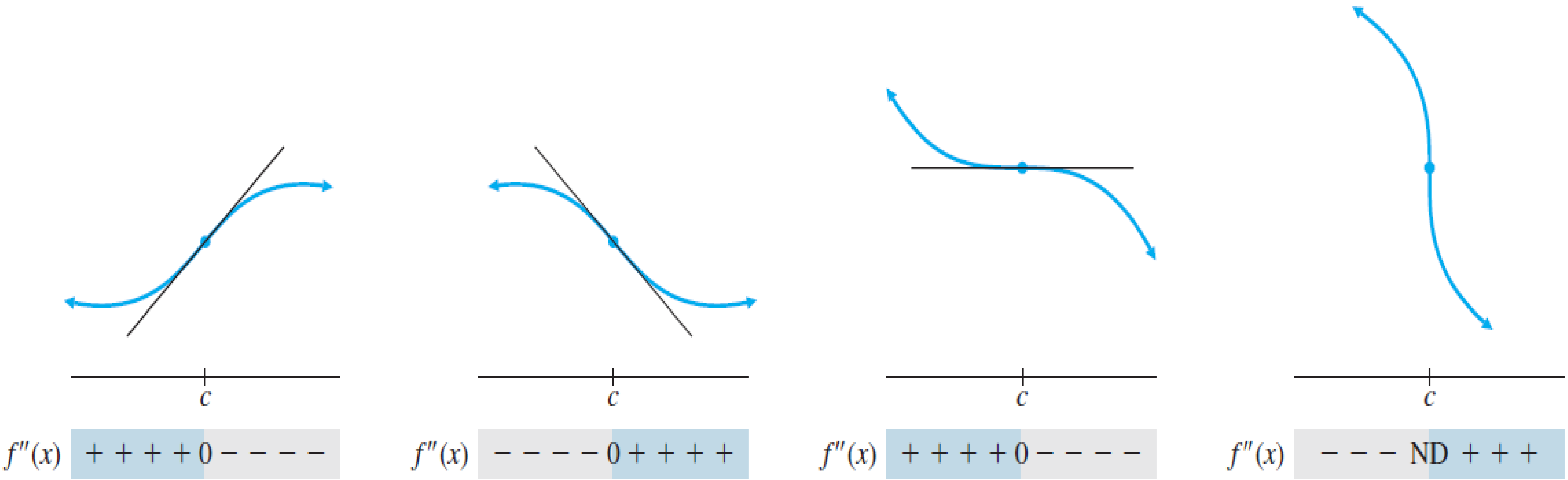
1. Does $f''(x)$ change sign at c?

2. Is c is in the domain of f?

If both answers are "yes", then $(c, f(c))$ is an inflection point of f. Figure 5 illustrates several typical cases.

If $f'(c)$ exists and $f''(x)$ changes sign at $x = c$, then the tangent line at an inflection point $(c, f(c))$ will always lie below the graph on the side that is concave upward and above the graph on the side that is concave downward (see Fig. 5A, B, and C).

Inflection Points



(A) $f'(c) > 0$ (B) $f'(c) < 0$ (C) $f'(c) = 0$ (D) $f'(c)$ is not defined

Figure 5 **Inflection points**

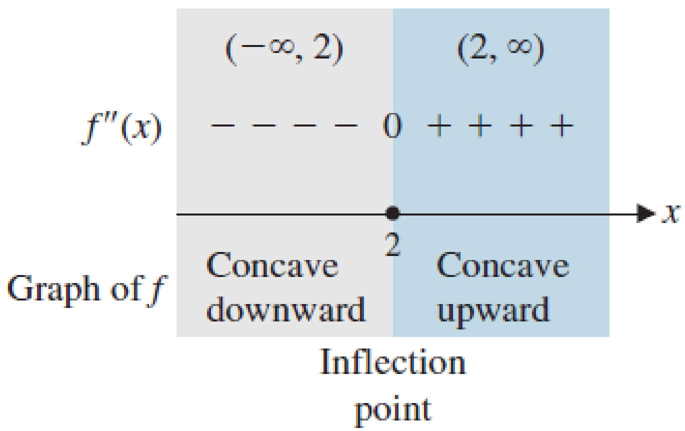
EXAMPLE 2 Locating Inflection Points Find the inflection point(s) of

$$f(x) = x^3 - 6x^2 + 9x + 1$$

SOLUTION Since inflection points occur at values of x where $f''(x)$ changes sign, we construct a sign chart for $f''(x)$.

$$\begin{aligned} f(x) &= x^3 - 6x^2 + 9x + 1 \\ f'(x) &= 3x^2 - 12x + 9 \\ f''(x) &= 6x - 12 = 6(x - 2) \end{aligned}$$

The sign chart for $f''(x) = 6(x - 2)$ (partition number is 2) is as follows:



Test Numbers	
x	$f''(x)$
1	-6 $(-)$
3	6 $(+)$

From the sign chart, we see that the graph of f has an inflection point at $x = 2$. That is, the point

$$(2, f(2)) = (2, 3) \qquad f(2) = 2^3 - 6 \cdot 2^2 + 9 \cdot 2 + 1 = 3$$

is an inflection point on the graph of f.

EXAMPLE 3 Locating Inflection Points Find the inflection point(s) of

$$f(x) = \ln(x^2 - 4x + 5)$$

SOLUTION First we find the domain of f. Since $\ln x$ is defined only for $x > 0$, f is defined only for

$$x^2 - 4x + 5 > 0 \quad \text{Complete the square (Section A.7).}$$
$$(x - 2)^2 + 1 > 0 \quad \text{True for all } x.$$

So the domain of f is $(-\infty, \infty)$. Now we find $f''(x)$ and construct a sign chart for it.

$$f(x) = \ln(x^2 - 4x + 5)$$

$$f'(x) = \frac{2x - 4}{x^2 - 4x + 5}$$

$$f''(x) = \frac{(x^2 - 4x + 5)(2x - 4)' - (2x - 4)(x^2 - 4x + 5)'}{(x^2 - 4x + 5)^2}$$

$$= \frac{(x^2 - 4x + 5)2 - (2x - 4)(2x - 4)}{(x^2 - 4x + 5)^2}$$

$$= \frac{2x^2 - 8x + 10 - 4x^2 + 16x - 16}{(x^2 - 4x + 5)^2}$$

$$= \frac{-2x^2 + 8x - 6}{(x^2 - 4x + 5)^2}$$

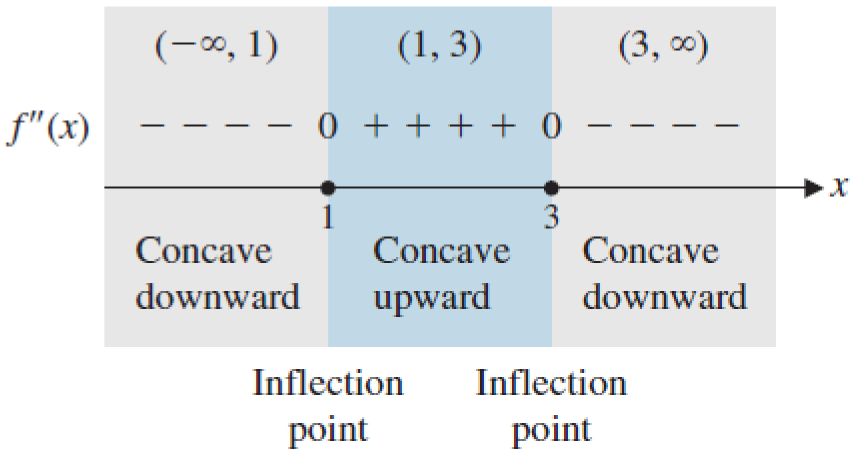
$$= \frac{-2(x - 1)(x - 3)}{(x^2 - 4x + 5)^2}$$

EXAMPLE 3 Locating Inflection Points Find the inflection point(s) of

$$f(x) = \ln(x^2 - 4x + 5)$$

SOLUTION

The partition numbers for $f''(x)$ are $x = 1$ and $x = 3$.
Sign chart for $f''(x)$:



Test Numbers

x	$f''(x)$
0	$-\frac{6}{25}$ (−)
2	2 (+)
4	$-\frac{6}{25}$ (−)

The sign chart shows that the graph of f has inflection points at $x = 1$ and $x = 3$.

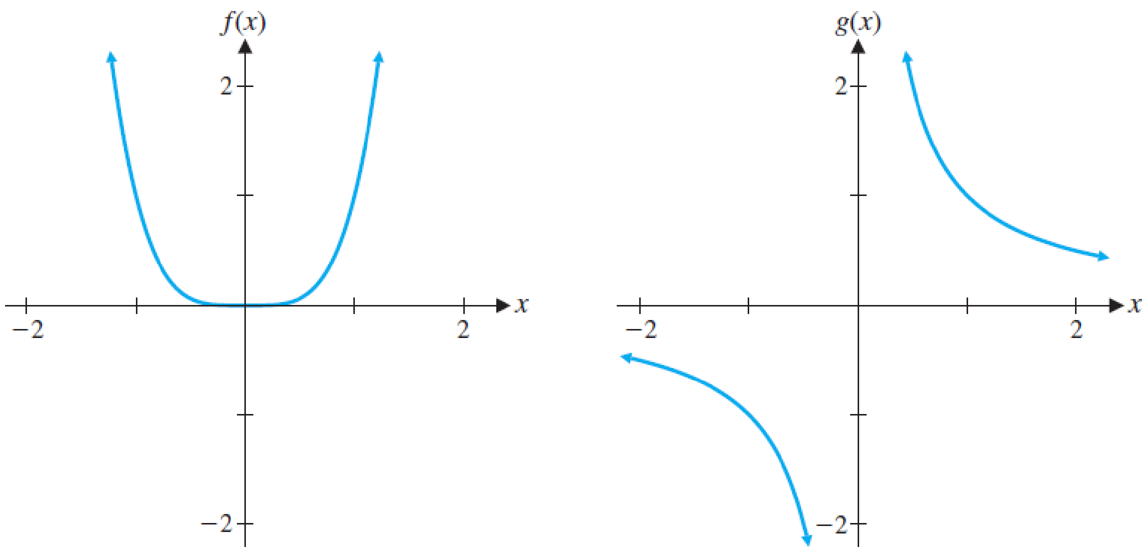
Since $f(1) = \ln 2$ and $f(3) = \ln 2$, the inflection points are $(1, \ln 2)$ and $(3, \ln 2)$.

CONCEPTUAL INSIGHT

It is important to remember that the partition numbers for f'' are only *candidates* for inflection points. The function f must be defined at $x = c$, and the second derivative must change sign at $x = c$ in order for the graph to have an inflection point at $x = c$. For example, consider

$$f(x) = x^4 \qquad g(x) = \frac{1}{x}$$
$$f'(x) = 4x^3 \qquad g'(x) = -\frac{1}{x^2}$$
$$f''(x) = 12x^2 \qquad g''(x) = \frac{2}{x^3}$$

In each case, $x = 0$ is a partition number for the second derivative, but neither the graph of $f(x)$ nor the graph of $g(x)$ has an inflection point at $x = 0$. Function f does not have an inflection point at $x = 0$ because $f''(x)$ does not change sign at $x = 0$ (see Fig. 6A). Function g does not have an inflection point at $x = 0$ because $g(0)$ is not defined (see Fig. 6B).



(A) $f(x) = x^4$ (B) $g(x) = \frac{1}{x}$

Figure 6

Point of Diminishing Returns

If a company decides to increase spending on advertising, it would expect sales to increase. At first, sales will increase at an increasing rate and then increase at a decreasing rate. The dollar amount x at which the rate of change of sales goes from increasing to decreasing is called the **point of diminishing returns**. This is also the amount at which the rate of change has a maximum value. Money spent beyond this amount may increase sales but at a lower rate.

EXAMPLE 7 Maximum Rate of Change Currently, a discount appliance store is selling 200 large-screen TVs monthly. If the store invests \$$x$ thousand in an advertising campaign, the ad company estimates that monthly sales will be given by

$$N(x) = 3x^3 - 0.25x^4 + 200 \qquad 0 \le x \le 9$$

When is the rate of change of sales increasing and when is it decreasing? What is the point of diminishing returns and the maximum rate of change of sales? Graph N and N' on the same coordinate system.

SOLUTION The rate of change of sales with respect to advertising expenditures is

$$N'(x) = 9x^2 - x^3 = x^2(9 - x)$$

To determine when $N'(x)$ is increasing and decreasing, we find $N''(x)$, the derivative of $N'(x)$:

$$N''(x) = 18x - 3x^2 = 3x(6 - x)$$

The information obtained by analyzing the signs of $N'(x)$ and $N''(x)$ is summarized in Table 2 (sign charts are omitted).

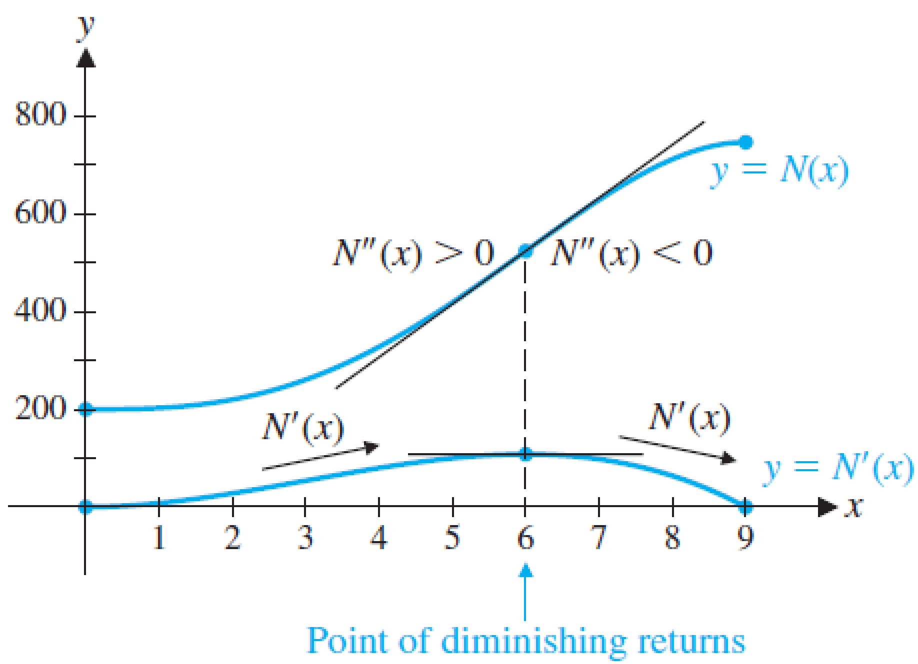
Table 2

x	$N''(x)$	$N'(x)$	$N'(x)$	$N(x)$
$0 < x < 6$	+	+	Increasing	Increasing, concave upward
$x = 6$	0	+	Local maximum	Inflection point
$6 < x < 9$	−	+	Decreasing	Increasing, concave downward

SOLUTION

Examining Table 2, we see that $N'(x)$ is increasing on $(0, 6)$ and decreasing on $(6, 9)$. The point of diminishing returns is $x = 6$ and the maximum rate of change is $N'(6) = 108$. Note that $N'(x)$ has a local maximum and $N(x)$ has an inflection point at $x = 6$ [the inflection point of $N(x)$ is $(6, 524)$].

So if the store spends $6,000 on advertising, monthly sales are expected to be 524 TVs, and sales are expected to increase at a rate of 108 TVs per thousand dollars spent on advertising. Money spent beyond the $6,000 would increase sales, but at a lower rate.

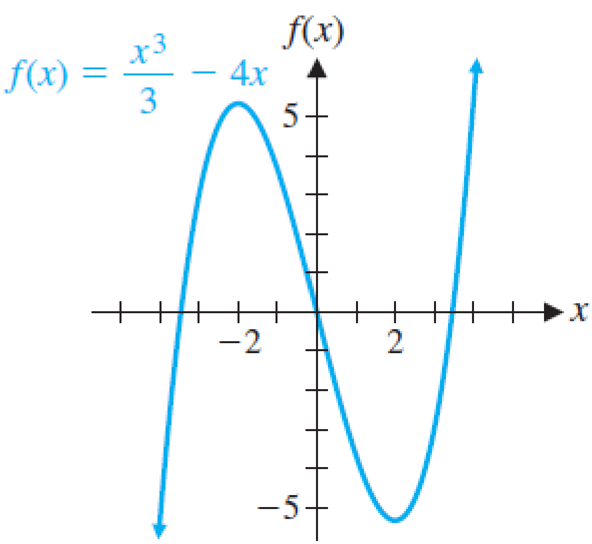


Absolute Maxima and Minima

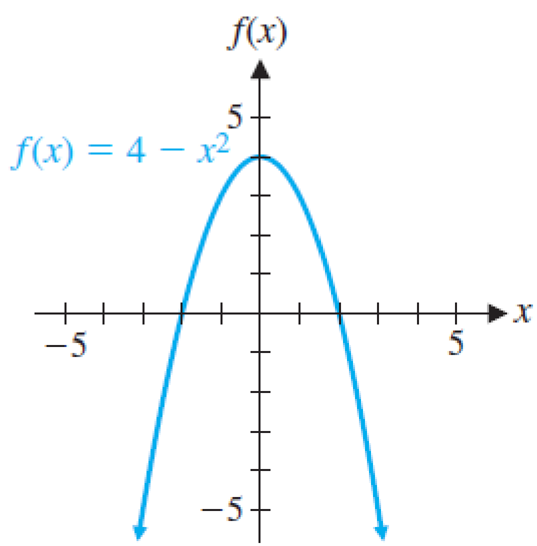
Recall that $f(c)$ is a local maximum if $f(x) \le f(c)$ for x near c and a local minimum if $f(x) \ge f(c)$ for x near c. Now we are interested in finding the largest and the smallest values of $f(x)$ throughout the domain of f.

DEFINITION Absolute Maxima and Minima

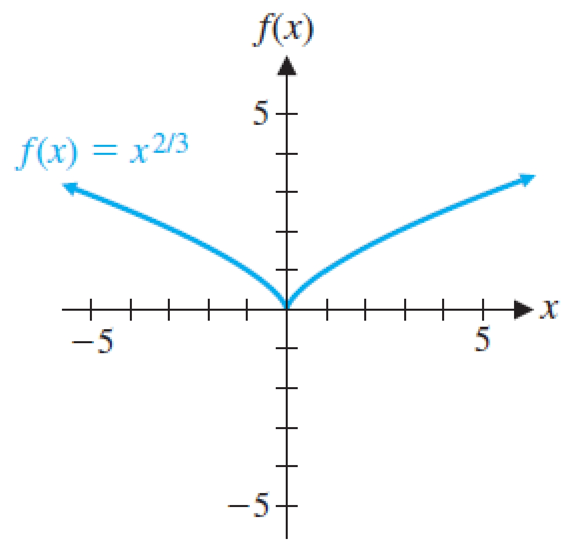
If $f(c) \ge f(x)$ for all x in the domain of f, then $f(c)$ is called the **absolute maximum** of f. If $f(c) \le f(x)$ for all x in the domain of f, then $f(c)$ is called the **absolute minimum** of f. An absolute maximum or absolute minimum is called an **absolute extremum.**



(A) No absolute maximum or minimum
$f(-2) = \frac{16}{3}$ is a local maximum
$f(2) = -\frac{16}{3}$ is a local minimum



(B) $f(0) = 4$ is the absolute maximum
No absolute minimum



(C) $f(0) = 0$ is the absolute minimum
No absolute maximum

Absolute Maxima and Minima

THEOREM 1 Extreme Value Theorem

A function f that is continuous on a closed interval $[a, b]$ has both an absolute maximum and an absolute minimum on that interval.

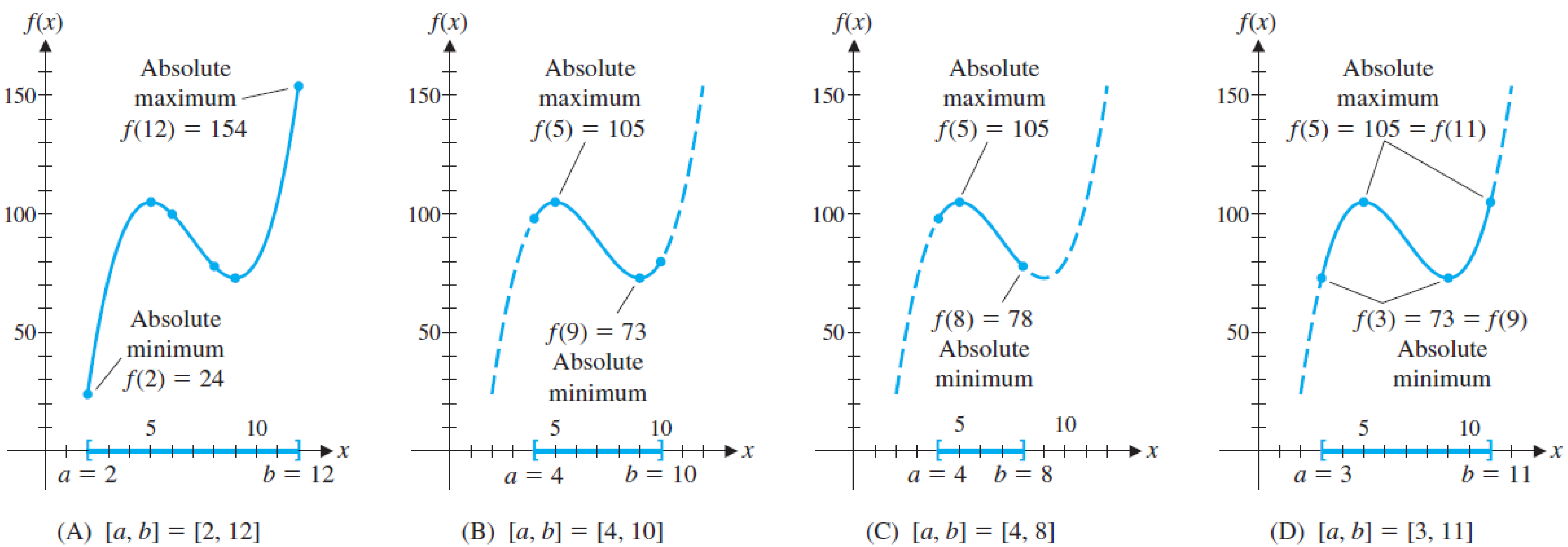


Figure 2 **Absolute extrema for $f(x) = x^3 - 21x^2 + 135x - 170$ on various closed intervals**

Absolute Maxima and Minima

THEOREM 2 Locating Absolute Extrema

Absolute extrema (if they exist) must occur at critical numbers or at endpoints.

PROCEDURE Finding Absolute Extrema on a Closed Interval

Step 1 Check to make certain that f is continuous over $[a, b]$.

Step 2 Find the critical numbers in the interval (a, b).

Step 3 Evaluate f at the endpoints a and b and at the critical numbers found in step 2.

Step 4 The absolute maximum of f on $[a, b]$ is the largest value found in step 3.

Step 5 The absolute minimum of f on $[a, b]$ is the smallest value found in step 3.

Absolute Maxima and Minima

EXAMPLE 1 Finding Absolute Extrema Find the absolute maximum and absolute minimum of

$$f(x) = x^3 + 3x^2 - 9x - 7$$

on each of the following intervals:

(A) $[-6, 4]$ (B) $[-4, 2]$ (C) $[-2, 2]$

SOLUTION

(A) The function is continuous for all values of x.

$$f'(x) = 3x^2 + 6x - 9 = 3(x - 1)(x + 3)$$

So, $x = -3$ and $x = 1$ are the critical numbers in the interval $(-6, 4)$. Evaluate f at the endpoints and critical numbers $(-6, -3, 1, \text{and } 4)$, and choose the largest and smallest values.

$f(-6) = -61$ Absolute minimum

$f(-3) = 20$

$f(1) = -12$

$f(4) = 69$ Absolute maximum

The absolute maximum of f on $[-6, 4]$ is 69, and the absolute minimum is -61.

Absolute Maxima and Minima

SOLUTION

(B) Interval: $[-4, 2]$

x	$f(x)$	
-4	13	
-3	20	Absolute maximum
1	-12	Absolute minimum
2	-5	

The absolute maximum of f on $[-4, 2]$ is 20, and the absolute minimum is -12.

(C) Interval: $[-2, 2]$

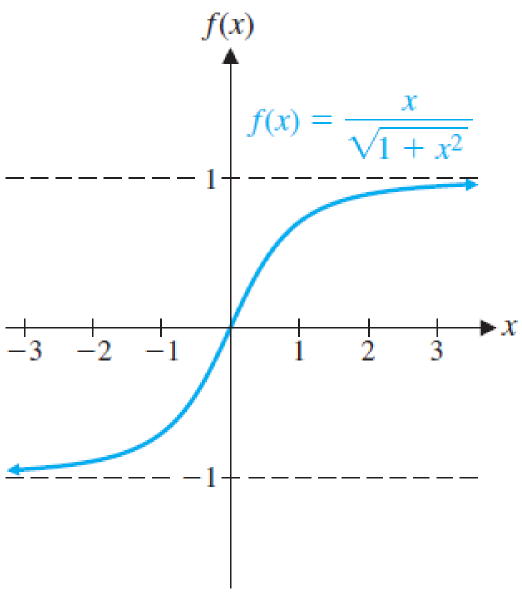
x	$f(x)$	
-2	15	Absolute maximum
1	-12	Absolute minimum
2	-5	

Note that the critical number $x = -3$ is not included in the table, because it is not in the interval $[-2, 2]$. The absolute maximum of f on $[-2, 2]$ is 15, and the absolute minimum is -12.

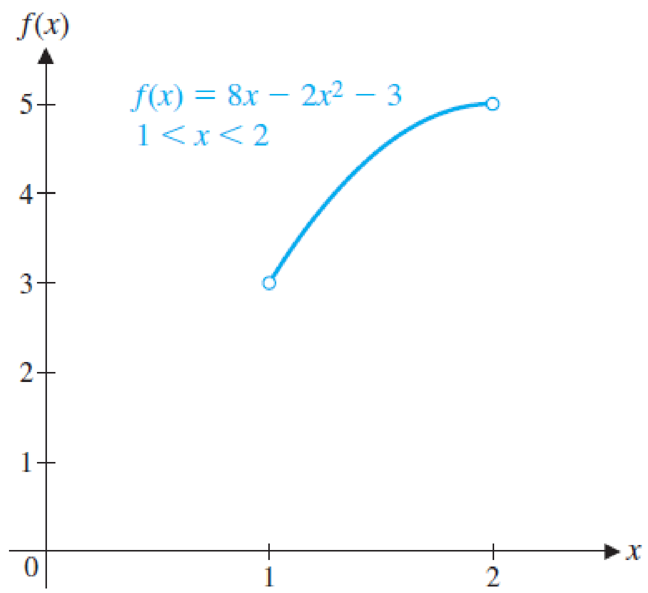
Absolute Maxima and Minima

Functions with no absolute extrema

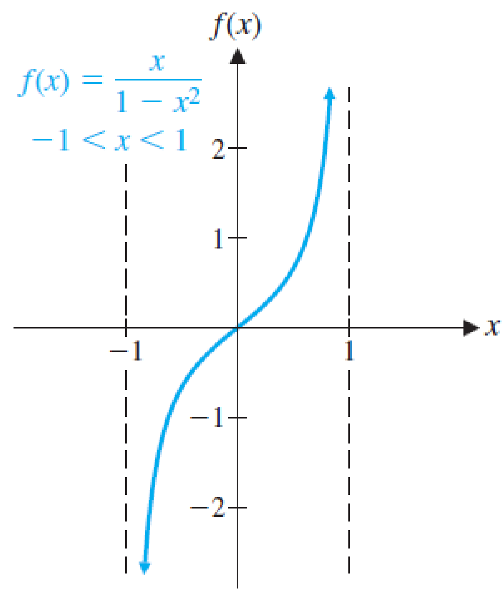
Now, suppose that we want to find the absolute maximum or minimum of a function that is continuous on an interval that is not closed. Since Theorem 1 no longer applies, we cannot be certain that the absolute maximum or minimum value exists.



(A) No absolute extrema on $(-\infty, \infty)$:
$-1 < f(x) < 1$ for all x
$[f(x) \neq 1$ or -1 for any $x]$



(B) No absolute extrema on $(1, 2)$:
$3 < f(x) < 5$ for $x \in (1, 2)$
$[f(x) \neq 3$ or 5 for any $x \in (1, 2)]$



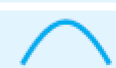
(C) No absolute extrema on $(-1, 1)$:
Graph has vertical asymptotes at $x = -1$ and $x = 1$

Second Derivative Test

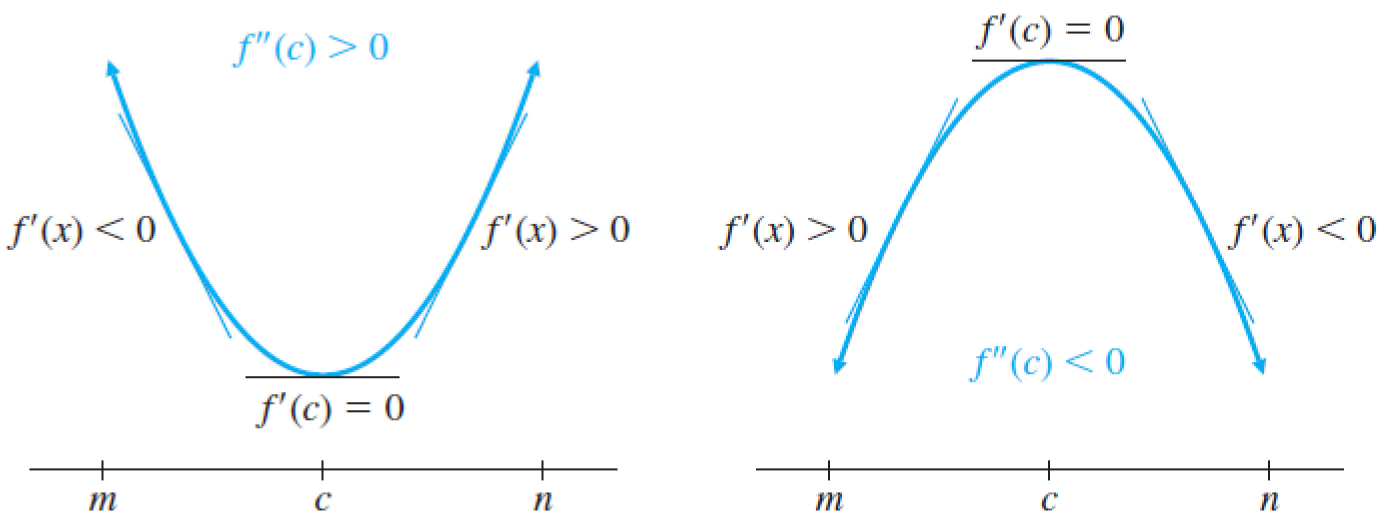
The second derivative can be used to classify the local extrema of a function. Suppose that f is a function satisfying $f'(c) = 0$ and $f''(c) > 0$. First, note that if $f''(c) > 0$, then it follows from the properties of limits* that $f''(x) > 0$ in some interval (m, n) containing c. Thus, the graph of f must be concave upward in this interval. But this implies that $f'(x)$ is increasing in the interval. Since $f'(c) = 0$, $f'(x)$ must change from negative to positive at $x = c$, and $f(c)$ is a local minimum (see Fig. 4). Reasoning in the same fashion, we conclude that if $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a local maximum. Of course, it is possible that both $f'(c) = 0$ and $f''(c) = 0$. In this case, the second derivative cannot be used to determine the shape of the graph around $x = c$; $f(c)$ may be a local minimum, a local maximum, or neither.

RESULT Second-Derivative Test for Local Extrema

Let c be a critical number of $f(x)$ such that $f'(c) = 0$. If the second derivative $f''(c) > 0$, then $f(c)$ is a local minimum. If $f''(c) < 0$, then $f(c)$ is a local maximum.

$f'(c)$	$f''(c)$	Graph of f is:	$f(c)$	Example
0	+	Concave upward	Local minimum	
0	−	Concave downward	Local maximum	
0	0	?	Test does not apply	

Second Derivative Test



(A) $f'(c) = 0$ and $f''(c) > 0$ implies $f(c)$ is a local minimum

(B) $f'(c) = 0$ and $f''(c) < 0$ implies $f(c)$ is a local maximum

Second Derivative Test

EXAMPLE 2 Testing Local Extrema Find the local maxima and minima for each function. Use the second-derivative test for local extrema when it applies.

(A) $f(x) = x^3 - 6x^2 + 9x + 1$

(B) $f(x) = xe^{-0.2x}$

(C) $f(x) = \frac{1}{6}x^6 - 4x^5 + 25x^4$

SOLUTION

(A) Find first and second derivatives and determine critical numbers:

$$f(x) = x^3 - 6x^2 + 9x + 1$$
$$f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$$
$$f''(x) = 6x - 12 = 6(x - 2)$$

Critical numbers are $x = 1$ and $x = 3$.

$f''(1) = -6 < 0$ f has a local maximum at $x = 1$.

$f''(3) = 6 > 0$ f has a local minimum at $x = 3$.

Substituting $x = 1$ in the expression for $f(x)$, we find that $f(1) = 5$ is a local maximum. Similarly, $f(3) = 1$ is a local minimum.

Second Derivative Test

EXAMPLE 2 Testing Local Extrema Find the local maxima and minima for each function. Use the second-derivative test for local extrema when it applies.

(A) $f(x) = x^3 - 6x^2 + 9x + 1$

(B) $f(x) = xe^{-0.2x}$

(C) $f(x) = \frac{1}{6}x^6 - 4x^5 + 25x^4$

SOLUTION

(B)

$$\begin{aligned} f(x) &= xe^{-0.2x} \\ f'(x) &= e^{-0.2x} + xe^{-0.2x}(-0.2) \\ &= e^{-0.2x}(1 - 0.2x) \\ f''(x) &= e^{-0.2x}(-0.2)(1 - 0.2x) + e^{-0.2x}(-0.2) \\ &= e^{-0.2x}(0.04x - 0.4) \end{aligned}$$

Critical number: $x = 1/0.2 = 5$

$$f''(5) = e^{-1}(-0.2) < 0$$ f has a local maximum at $x = 5$.

So $f(5) = 5e^{-0.2(5)} \approx 1.84$ is a local maximum.

Second Derivative Test

SOLUTION

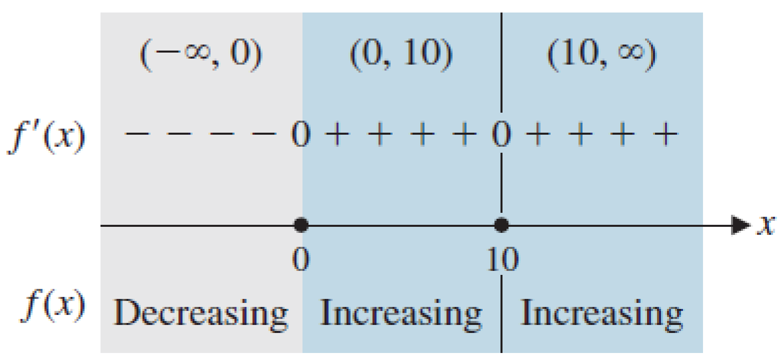
(C)

$$f(x) = \tfrac{1}{6}x^6 - 4x^5 + 25x^4$$
$$f'(x) = x^5 - 20x^4 + 100x^3 = x^3(x - 10)^2$$
$$f''(x) = 5x^4 - 80x^3 + 300x^2$$

Critical numbers are $x = 0$ and $x = 10$.

$f''(0) = 0$ The second-derivative test fails at both critical numbers, so
$f''(10) = 0$ the first-derivative test must be used.

Sign chart for $f'(x) = x^3(x - 10)^2$ (partition numbers for f' are 0 and 10):



Test Numbers

x	$f'(x)$
-1	-121 $(-)$
1	81 $(+)$
11	$1{,}331$ $(+)$

From the chart, we see that $f(x)$ has a local minimum at $x = 0$ and does not have a local extremum at $x = 10$. So $f(0) = 0$ is a local minimum.

Second Derivative Test

CONCEPTUAL INSIGHT

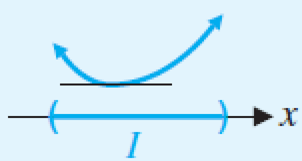
The second-derivative test for local extrema does not apply if $f''(c) = 0$ or if $f''(c)$ is not defined. As Example 2C illustrates, if $f''(c) = 0$, then $f(c)$ may or may not be a local extremum. Some other method, such as the first-derivative test, must be used when $f''(c) = 0$ or $f''(c)$ does not exist.

The solution of many optimization problems involves searching for an absolute extremum. If the function in question has only one critical number, then the second-derivative test for local extrema not only classifies the local extremum but also guarantees that the local extremum is, in fact, the absolute extremum.

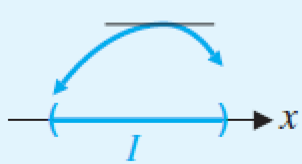
THEOREM 3 Second-Derivative Test for Absolute Extrema on an Open Interval

Let f be continuous on an open interval I with only one critical number c in I.

If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is the absolute minimum of f on I.



If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is the absolute maximum of f on I.



Second Derivative Test

EXAMPLE 3 Finding Absolute Extrema on an Open Interval Find the absolute extrema of each function on $(0, \infty)$.

(A) $f(x) = x + \frac{4}{x}$ (B) $f(x) = (\ln x)^2 - 3 \ln x$

SOLUTION

(A) $f(x) = x + \frac{4}{x}$

$$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2} = \frac{(x - 2)(x + 2)}{x^2}$$

Critical numbers are $x = -2$ and $x = 2$.

$$f''(x) = \frac{8}{x^3}$$

The only critical number in the interval $(0, \infty)$ is $x = 2$. Since $f''(2) = 1 > 0$, $f(2) = 4$ is the absolute minimum of f on $(0, \infty)$.

Second Derivative Test

EXAMPLE 3 Finding Absolute Extrema on an Open Interval Find the absolute extrema of each function on $(0, \infty)$.

(A) $f(x) = x + \frac{4}{x}$ (B) $f(x) = (\ln x)^2 - 3 \ln x$

SOLUTION

(B) $f(x) = (\ln x)^2 - 3 \ln x$

$$f'(x) = (2 \ln x)\frac{1}{x} - \frac{3}{x} = \frac{2 \ln x - 3}{x}$$

Critical number is $x = e^{3/2}$.

$$f''(x) = \frac{x\frac{2}{x} - (2 \ln x - 3)}{x^2} = \frac{5 - 2 \ln x}{x^2}$$

The only critical number in the interval $(0, \infty)$ is $x = e^{3/2}$. Since $f''(e^{3/2}) = 2/e^3 > 0$, $f(e^{3/2}) = -2.25$ is the absolute minimum of f on $(0, \infty)$.

Matched Problem 3 Find the absolute extrema of each function on $(0, \infty)$.

(A) $f(x) = 12 - x - \frac{5}{x}$ (B) $f(x) = 5 \ln x - x$

Optimization Problems

Optimization

PROCEDURE Strategy for Solving Optimization Problems

Step 1 Introduce variables, look for relationships among the variables, and construct a mathematical model of the form

$$\text{Maximize (or minimize) } f(x) \text{ on the interval } I$$

Step 2 Find the critical numbers of $f(x)$.

Step 3 Use the procedures developed in Section 12.5 to find the absolute maximum (or minimum) of $f(x)$ on the interval I and the numbers x where this occurs.

Step 4 Use the solution to the mathematical model to answer all the questions asked in the problem.

Maximizing Revenue and Profit

EXAMPLE 3 Maximizing Revenue An office supply company sells x permanent markers per year at \$$p$ per marker. The price–demand equation for these markers is $p = 10 - 0.001x$. What price should the company charge for the markers to maximize revenue? What is the maximum revenue?

SOLUTION

$$\begin{aligned} \text{Revenue} &= \text{price} \times \text{demand} \\ R(x) &= (10 - 0.001x)x \\ &= 10x - 0.001x^2 \end{aligned}$$

Both price and demand must be nonnegative, so

$$x \ge 0 \quad \text{and} \quad p = 10 - 0.001x \ge 0$$
$$10 \ge 0.001x$$
$$10{,}000 \ge x$$

The mathematical model for this problem is

$$\text{Maximize} \quad R(x) = 10x - 0.001x^2 \qquad 0 \le x \le 10{,}000$$
$$R'(x) = 10 - 0.002x$$
$$10 - 0.002x = 0$$
$$10 = 0.002x$$
$$x = \frac{10}{0.002} = 5{,}000 \qquad \text{Critical number}$$

Maximizing Revenue and Profit

EXAMPLE 3 Maximizing Revenue An office supply company sells x permanent markers per year at $\$p$ per marker. The price–demand equation for these markers is $p = 10 - 0.001x$. What price should the company charge for the markers to maximize revenue? What is the maximum revenue?

SOLUTION

Use the second-derivative test for absolute extrema:

$$R''(x) = -0.002 < 0 \quad \text{for all } x$$

$$\text{Max } R(x) = R(5{,}000) = \$25{,}000$$

When the demand is $x = 5{,}000$, the price is

$$10 - 0.001(5{,}000) = \$5 \qquad p = 10 - 0.001x$$

The company will realize a maximum revenue of \$25,000 when the price of a marker is \$5.

Maximizing Revenue and Profit

EXAMPLE 4 Maximizing Profit The total annual cost of manufacturing x permanent markers for the office supply company in Example 3 is

$$C(x) = 5{,}000 + 2x$$

What is the company's maximum profit? What should the company charge for each marker, and how many markers should be produced?

SOLUTION Using the revenue model in Example 3, we have

$$\begin{aligned} \text{Profit} &= \text{Revenue} - \text{Cost} \\ P(x) &= R(x) - C(x) \\ &= 10x - 0.001x^2 - 5{,}000 - 2x \\ &= 8x - 0.001x^2 - 5{,}000 \end{aligned}$$

The mathematical model for profit is

$$\text{Maximize} \quad P(x) = 8x - 0.001x^2 - 5{,}000 \qquad 0 \le x \le 10{,}000$$

The restrictions on x come from the revenue model in Example 3.

$$\begin{aligned} P'(x) = 8 - 0.002x &= 0 \\ 8 &= 0.002x \\ x &= \frac{8}{0.002} = 4{,}000 \end{aligned}$$

Critical number

Maximizing Revenue and Profit

SOLUTION

$$P''(x) = -0.002 < 0 \quad \text{for all } x$$

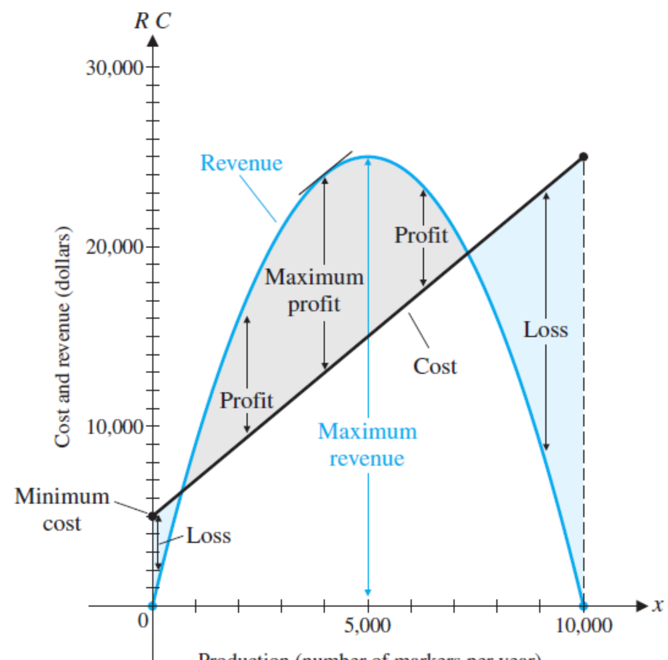
Since $x = 4{,}000$ is the only critical number and $P''(x) < 0$,

$$\text{Max } P(x) = P(4{,}000) = \$11{,}000$$

Using the price–demand equation from Example 3 with $x = 4{,}000$, we find that

$$p = 10 - 0.001(4{,}000) = \$6 \qquad p = 10 - 0.001x$$

A maximum profit of $11,000 is realized when 4,000 markers are manufactured annually and sold for $6 each.



Maximizing Revenue and Profit

EXAMPLE 5 Maximizing Profit The government decides to tax the company in Example 4 $2 for each marker produced. Taking into account this additional cost, how many markers should the company manufacture annually to maximize its profit? What is the maximum profit? How much should the company charge for the markers to realize the maximum profit?

SOLUTION The tax of $2 per unit changes the company's cost equation:

$$\begin{aligned} C(x) &= \text{original cost} + \text{tax} \\ &= 5{,}000 + 2x + 2x \\ &= 5{,}000 + 4x \end{aligned}$$

The new profit function is

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 10x - 0.001x^2 - 5{,}000 - 4x \\ &= 6x - 0.001x^2 - 5{,}000 \end{aligned}$$

Maximizing Revenue and Profit

SOLUTION

So, we must solve the following equation:

$$\text{Maximize} \quad P(x) = 6x - 0.001x^2 - 5{,}000 \qquad 0 \le x \le 10{,}000$$

$$P'(x) = 6 - 0.002x$$

$$6 - 0.002x = 0$$

$$x = 3{,}000 \quad \text{Critical number}$$

$$P''(x) = -0.002 < 0 \quad \text{for all } x$$

$$\text{Max } P(x) = P(3{,}000) = \$4{,}000$$

Using the price–demand equation (Example 3) with $x = 3{,}000$, we find that

$$p = 10 - 0.001(3{,}000) = \$7 \qquad p = 10 - 0.001x$$

The company's maximum profit is \$4,000 when 3,000 markers are produced and sold annually at a price of \$7.

Even though the tax caused the company's cost to increase by \$2 per marker, the price that the company should charge to maximize its profit increases by only \$1. The company must absorb the other \$1, with a resulting decrease of \$7,000 in maximum profit.

Maximizing Revenue and Profit

EXAMPLE 6 Maximizing Revenue When a management training company prices its seminar on management techniques at $400 per person, 1,000 people will attend the seminar. The company estimates that for each $5 reduction in price, an additional 20 people will attend the seminar. How much should the company charge for the seminar in order to maximize its revenue? What is the maximum revenue?

SOLUTION Let x represent the number of $5 price reductions.

$$400 - 5x = \text{price per customer}$$
$$1{,}000 + 20x = \text{number of customers}$$
$$\text{Revenue} = (\text{price per customer})(\text{number of customers})$$
$$R(x) = (400 - 5x) \times (1{,}000 + 20x)$$

Since price cannot be negative, we have

$$400 - 5x \geq 0$$
$$400 \geq 5x$$
$$80 \geq x \qquad \text{or} \qquad x \leq 80$$

Table 2

x	$R(x)$
0	400,000
15	422,500
80	0

Maximizing Revenue and Profit

EXAMPLE 6 Maximizing Revenue When a management training company prices its seminar on management techniques at \$400 per person, 1,000 people will attend the seminar. The company estimates that for each \$5 reduction in price, an additional 20 people will attend the seminar. How much should the company charge for the seminar in order to maximize its revenue? What is the maximum revenue?

SOLUTION

A negative value of x would result in a price increase. Since the problem is stated in terms of price reductions, we must restrict x so that $x \geq 0$. Putting all this together, we have the following model:

$$\text{Maximize} \quad R(x) = (400 - 5x)(1{,}000 + 20x) \quad \text{for } 0 \leq x \leq 80$$

$$R(x) = 400{,}000 + 3{,}000x - 100x^2$$

$$R'(x) = 3{,}000 - 200x = 0$$

$$3{,}000 = 200x$$

$$x = 15 \qquad \text{Critical number}$$

Since $R(x)$ is continuous on the interval [0, 80], we can determine the behavior of the graph by constructing a table. Table 2 shows that $R(15) = \$422{,}500$ is the absolute maximum revenue. The price of attending the seminar at $x = 15$ is $400 - 5(15) = \$325$. The company should charge \$325 for the seminar in order to receive a maximum revenue of \$422,500.

Maximizing Revenue and Profit

EXAMPLE 7 Maximizing Revenue After additional analysis, the management training company in Example 6 decides that its estimate of attendance was too high. Its new estimate is that only 10 additional people will attend the seminar for each \$5 decrease in price. All other information remains the same. How much should the company charge for the seminar now in order to maximize revenue? What is the new maximum revenue?

SOLUTION Under the new assumption, the model becomes

$$\text{Maximize} \quad R(x) = (400 - 5x)(1{,}000 + 10x) \qquad 0 \le x \le 80$$
$$= 400{,}000 - 1{,}000x - 50x^2$$
$$R'(x) = -1{,}000 - 100x = 0$$
$$-1{,}000 = 100x$$
$$x = -10 \qquad \text{Critical number}$$

Table 3

x	$R(x)$
0	400,000
80	0

Note that $x = -10$ is not in the interval [0, 80]. Since $R(x)$ is continuous on [0, 80], we can use a table to find the absolute maximum revenue. Table 3 shows that the maximum revenue is $R(0) = \$400{,}000$. The company should leave the price at \$400. Any \$5 decreases in price will lower the revenue.