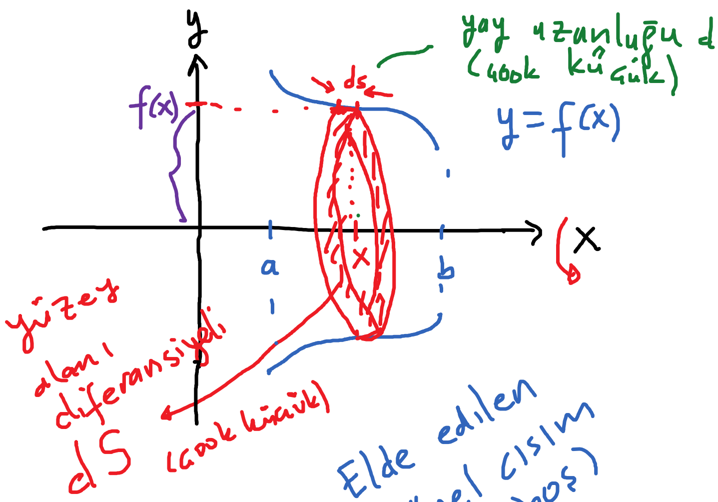
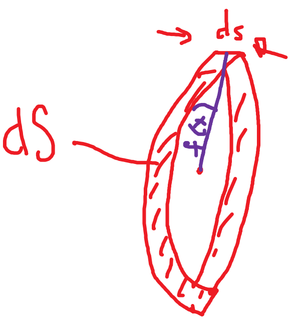
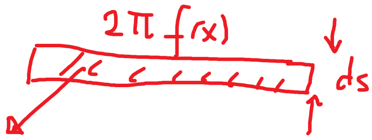
YÜZEY ALANI

Amaç: Dönel cisimlerin yüzey alanını hesaplamak

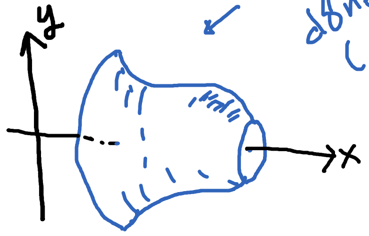




ds: f(x) yarıçaplı ds kalınlığında bir yüzük



Elde edilen dönel cisim (içi boş)



yüzey alanı diferansiyeli → $dS = 2\pi f(x)\, ds$

$$= 2\pi f(x) \sqrt{1+[f'(x)]^2}\, dx$$

Yüzey Alanı: $S = \int_a^b dS = \boxed{2\pi \int_a^b f(x) \sqrt{1+[f'(x)]^2}\, dx}$

Benzer şekilde, y-ekseni etrafında bir $x=g(y)$ eğrisi döndürüldüğünde $c \le y \le d$ için elde edilen cismin yüzey alanı

$$S = \int_c^d dS = 2\pi \int_c^d |g(y)| \sqrt{1+[g'(y)]^2}\, dy$$

olur. Veya, $y=f(x)$ ve $x \in [a,b]$ için

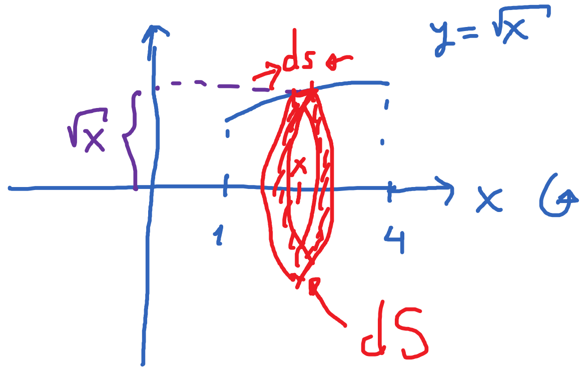
$$S = \int_a^b dS = 2\pi \int_a^b x \sqrt{1+[f'(x)]^2}\, dx$$

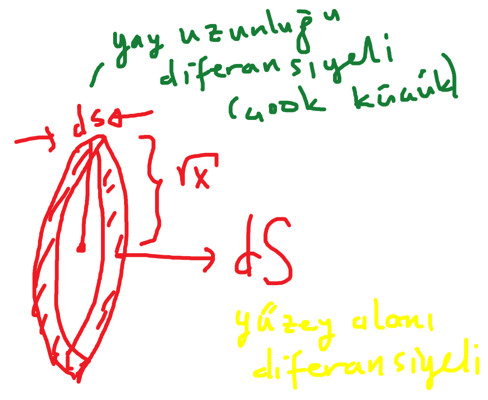
olur.

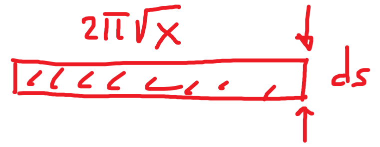
Fikir: dS'yi oluştur (diferansiyel kalkülüs) dx, dy, ds, dA → dS'leri hesapla (klasik matematik) → dS leri topla (integral kalkülüs) $\int dx, \int dy, \ldots$

ÖRN 1. $[1,4]$ aralığı üzerinde $y=\sqrt{x}$ in grafiğinin x-ekseni etrafında döndürülmesiyle oluşan cismin yüzey alanını bulunuz.

GÖZ.







$$dS = 2\pi\sqrt{x}\,ds$$

ds yi bulalım;

$$y=\sqrt{x} \Rightarrow \sqrt{1+(y')^2} = \sqrt{\left(\frac{1}{2\sqrt{x}}\right)^2+1} = \sqrt{\frac{1}{4x}+1}$$

O halde,

$$ds = \sqrt{1+(y')^2}\,dx = \frac{\sqrt{4x+1}}{2\sqrt{x}}\,dx$$

olur. Buradan,

$$dS = 2\pi\sqrt{x}\,ds = \cancel{2}\pi\cancel{\sqrt{x}}\underbrace{\frac{\sqrt{4x+1}}{\cancel{2\sqrt{x}}}\,dx}_{ds} = \pi\sqrt{4x+1}\,dx$$

bulunur. Sonuç olarak

$$S = \int_1^4 dS = \pi\int_1^4 \sqrt{4x+1}\,dx = \pi\int_5^{17} u^{1/2}\,\frac{du}{4}$$

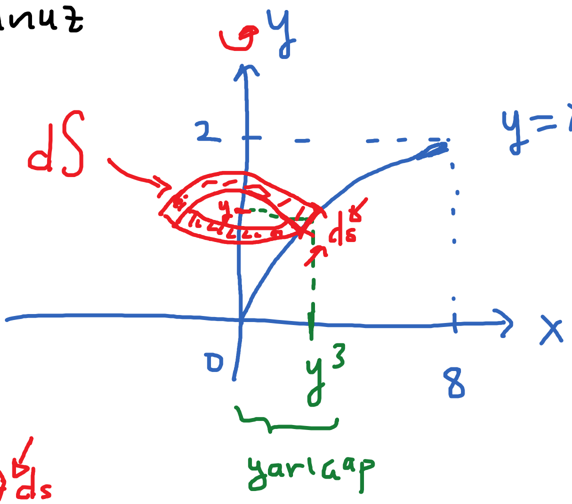
$u=4x+1$
$du=4\,dx$
$x=1 \Rightarrow u=5$
$x=4 \Rightarrow u=17$

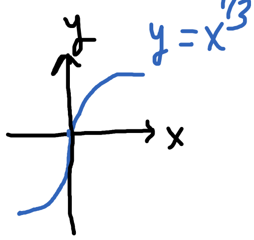
$$= \frac{\pi}{4}\,\frac{u^{3/2}}{\frac{3}{2}}\Bigg|_5^{17}$$

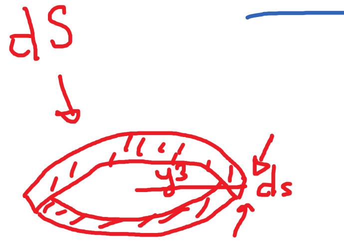
$$= \frac{\pi}{6}\left[17^{3/2}-5^{3/2}\right]$$

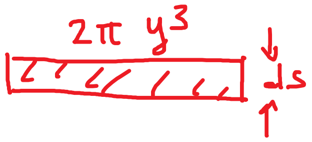
ÖRN 2. [0, 8] aralığı üzerinde $y = x^{1/3}$ ün grafiğinin y-ekseni etrafında döndürülmesiyle oluşan S yüzey alanını bulunuz

Çöz









$$dS = 2\pi y^3 \, ds$$

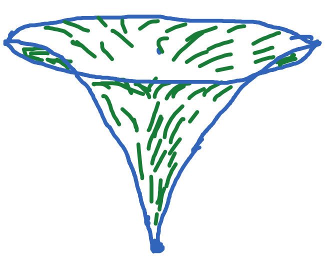
ds yi bulalım; $x = y^3$; $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$

$\frac{dx}{dy} = 3y^2$ $\qquad ds = \sqrt{1 + 9y^4} \, dy$

$$S = \int_0^2 dS = 2\pi \int_0^2 y^3 \, ds = 2\pi \int_0^2 \sqrt{1 + 9y^4} \, y^3 \, dy$$

$$\left(\begin{array}{l} u = 1 + 9y^4 \\ du = 36 y^3 dy \\ y = 0 \Rightarrow u = 1 \\ y = 2 \Rightarrow u = 145 \end{array} \right) = 2\pi \int_1^{145} u^{1/2} \frac{du}{36}$$

$$= \frac{\pi}{27} \left(145^{3/2} - 1 \right)$$

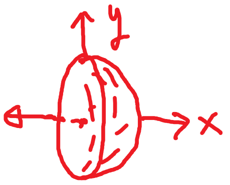


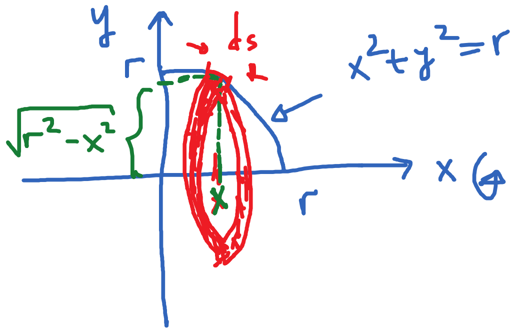
← Elde edilen yüzey

Not. Bu soru x değişken olacak şekilde de çözülebilir. Dene!

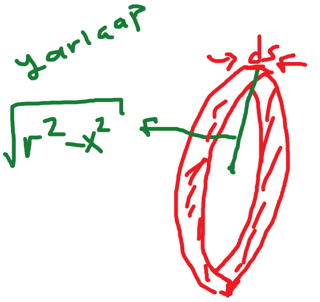
ÖRN 3 r yarıçaplı kürenin yüzey alanını bulunuz

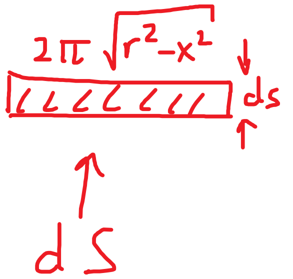
Çöz.





$x^2+y^2=r^2 \quad \left(y=\sqrt{r^2-x^2} \right)$





$dS = 2\pi\sqrt{r^2-x^2}\,ds$

ds yi bulalım. $\qquad y=\sqrt{r^2-x^2}=(r^2-x^2)^{1/2}$

$$y' = \frac{1}{2}(r^2-x^2)^{-1/2}\cdot(-2x) = -\frac{x}{\sqrt{r^2-x^2}}$$

$$\sqrt{1+(y')^2} = \sqrt{\frac{x^2}{r^2-x^2}+1} = \sqrt{\frac{r^2}{r^2-x^2}} = \frac{r}{\sqrt{r^2-x^2}}$$

$ds = \dfrac{r}{\sqrt{r^2-x^2}}\,dx$ olur

$$A(S) = 2\int_0^r dS = 2\int_0^r \underbrace{2\pi\sqrt{r^2-x^2}\,ds}_{dS}$$

$$= 4\pi\int_0^r \cancel{\sqrt{r^2-x^2}}\;\underbrace{\frac{r}{\cancel{\sqrt{r^2-x^2}}}\,dx}_{ds}$$

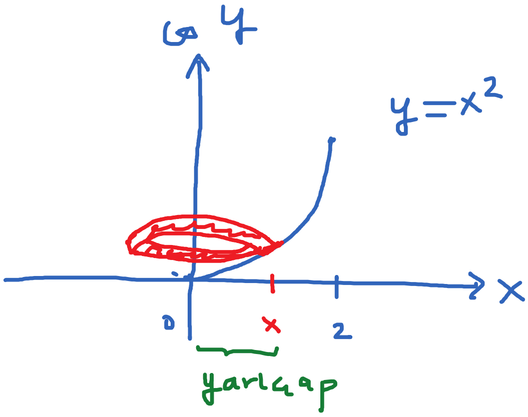
$$= 4\pi\int_0^r r\,dx = 4\pi r\,x\Big|_{x=0}^{x=r}$$

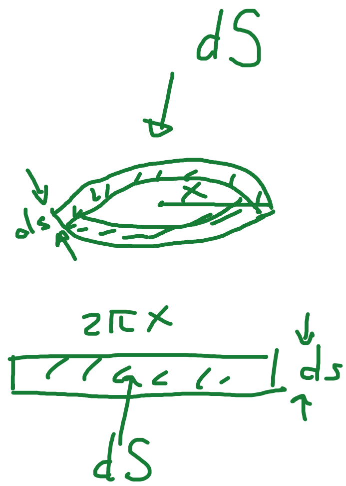
$$= 4\pi r^2$$

bulunur.

ÖRN 4 $[0,2]$ aralığında, $y=x^2$ parabolünün y-ekseni etrafında döndürülmesiyle elde edilen cismin yüzey alanını bulunuz

Çöz.





$$dS = 2\pi x \, ds$$

$$ds = \sqrt{1+(y')^2}\,dx = \sqrt{1+4x^2}\,dx$$

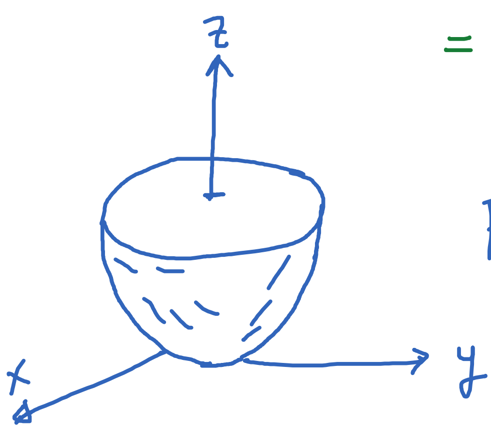
$$dS = 2\pi x \sqrt{1+4x^2}\,dx$$

$$A(S) = \int_0^2 dS = 2\pi \int_0^2 x\sqrt{1+4x^2}\,dx$$

$$= 2\pi \int_1^{17} \sqrt{u}\,\frac{du}{8}$$

$$\left(\begin{array}{ll} u = 1+4x^2, & x=0 \Rightarrow u=1 \\ du = 8x\,dx, & x=2 \Rightarrow u=17 \end{array}\right)$$

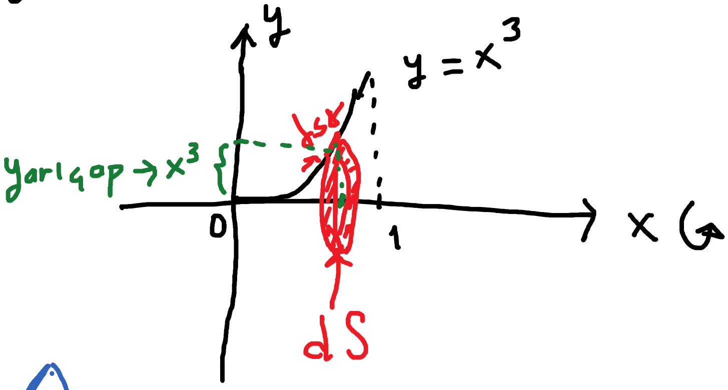
$$= \frac{\pi}{4} \left.\frac{u^{3/2}}{\frac{3}{2}}\right|_1^{17} = \frac{\pi}{6}\left(17^{3/2}-1\right)$$

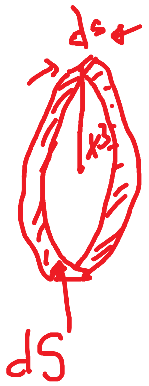


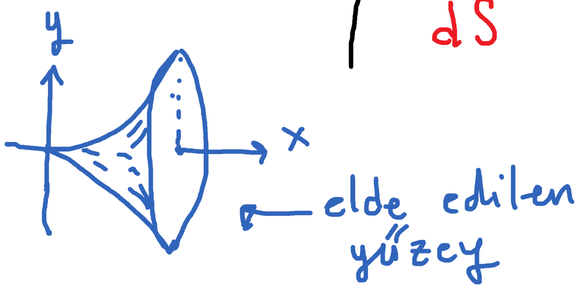
Paraboloid //

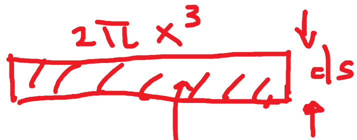
ÖRN. $[0,1]$ aralığı üzerinde $y=x^3$ eğrisinin x-ekseni etrafında döndürülmesiyle elde edilen cismin yüzey alanını bulunuz.

Çöz.









$$dS = 2\pi x^3\, ds$$

ds'yi hesaplayalım: $y=x^3$

$$\sqrt{1+(y')^2} = \sqrt{(3x^2)^2+1} = \sqrt{9x^4+1}$$

$$ds = \sqrt{9x^4+1}\, dx$$

O halde, $\quad dS = 2\pi x^3 \sqrt{9x^4+1}\, dx \quad$ olur.

$$S = \int_0^1 dS = 2\pi \int_0^1 x^3 \sqrt{9x^4+1}\, dx$$

$u = 9x^4+1$

$du = 36x^3\, dx$

$x=0 \Rightarrow u=1$

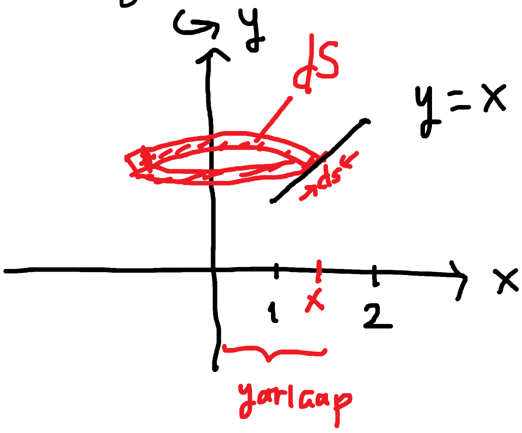
$x=1 \Rightarrow u=10$

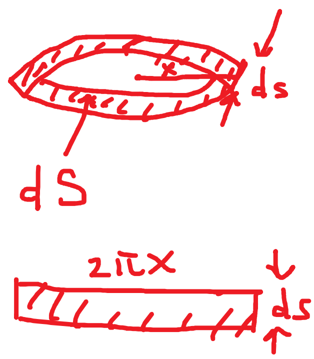
$$= 2\pi \int_1^{10} \sqrt{u}\, \frac{du}{36}$$

$$= \frac{\pi}{18} \left. \frac{u^{3/2}}{3/2} \right|_1^{10} = \frac{\pi}{27}\left(10^{3/2}-1\right) //$$

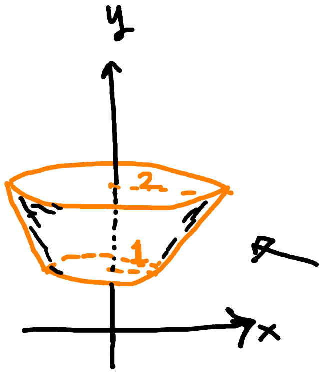
ÖRN. $[1,2]$ aralığında $y=x$ fonksiyonunun y-ekseni döndürülmesiyle elde edilen cismin yüzey alanını bulun.

ÇÖZ





$$dS = 2\pi x \, ds$$



elde edilen yüzey (içi boş)

ds 'yi bulalım: $\quad y=x$

$$\sqrt{1+(y')^2} = \sqrt{(1)^2+1} = \sqrt{2}$$

Yani, $\quad ds = \sqrt{2}\, dx$.

ds, dx, dx $\Rightarrow ds = \sqrt{2}\, dx$

$$dS = 2\pi x \, ds = 2\pi\sqrt{2}\, x \, dx$$

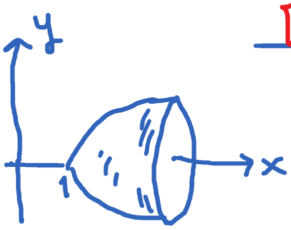
$$S = \int_1^2 dS = 2\pi\sqrt{2}\int_1^2 x\,dx = \cancel{2}\pi\sqrt{2}\,\frac{x^2}{\cancel{2}}\Big|_1^2$$

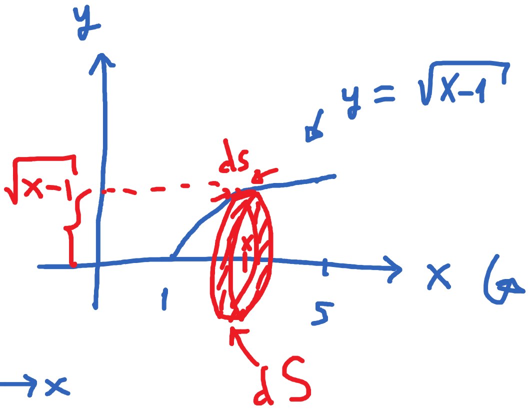
$$= \pi\sqrt{2}\,(4-1)$$

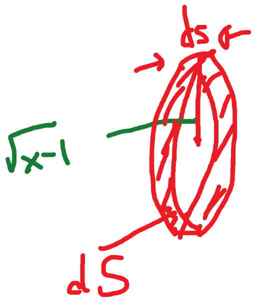
$$= 3\pi\sqrt{2} \;//$$

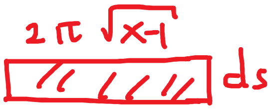
ÖRN. $y=\sqrt{x-1}$ eğrisinin $1\le x\le 5$ aralığındaki parçasının a) x-ekseni, b) y-ekseni etrafında döndürülmesiyle elde edilen cismin yüzey alanını belirli integral ile ifade ediniz.

a)









$dS = 2\pi\sqrt{x-1}\,ds$

$ds = ?$ $\quad y=\sqrt{x-1} \Rightarrow \sqrt{1+(y')^2}=\sqrt{\left(\frac{1}{2\sqrt{x-1}}\right)^2+1}$

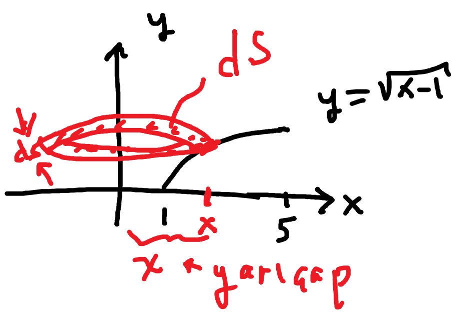
$ds=\sqrt{\frac{1}{4(x-1)}+1}\;dx$; $dS = 2\pi\sqrt{x-1}\sqrt{\frac{1}{4(x-1)}+1}\,dx$

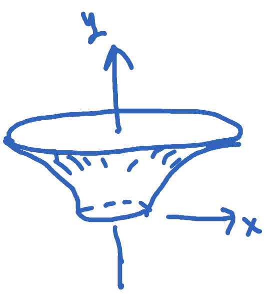
$= \cancel{2}\pi\cancel{\sqrt{x-1}}\,\frac{\sqrt{4x-3}}{\cancel{2}\cancel{\sqrt{x-1}}}\,dx$

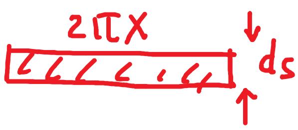
$= \pi\sqrt{4x-3}\,dx$

$S=\int_1^5 dS = 2\int_1^5\sqrt{4x-3}\,dx$ //

b)







$dS = 2\pi x\,ds$

$ds = ?$ $\quad y=\sqrt{x-1}$ $\quad ds=\frac{\sqrt{4x-3}}{2\sqrt{x-1}}\,dx$

$dS = 2\pi x\,\frac{\sqrt{4x-3}}{2\sqrt{x-1}}\,dx = \pi\,x\sqrt{\frac{4x-3}{x-1}}\,dx$

$S=\pi\int_1^5 x\sqrt{\frac{4x-3}{x-1}}\,dx$ elde edilir.

<u>ÖRN</u>. $x = 2\sqrt{y-1}$ eğrisinin $1 \leq y \leq 4$ aralığındaki parçasının y-ekseni etrafında döndürülmesiyle oluşan cismin yüzey alanını hesaplayınız.

<u>ÖDV</u>.

ÖRN [1, 8] aralığında, $y = x^{2/3}$ ün $y=4$ doğrusu etrafında döndürülmesiyle oluşan yüzey alanını veren belirli integrali ifade ediniz.

GÖZ. ÖDV.