Instructions: Show all of your work, and clearly indicate your answers.

Q1 Find the slope of the tangent line to the curve

$$f(x) = \sqrt{x}$$

at the point $x_0 = 1$ and then determine the equation of the tangent line.

Solution:

Compute: $m_{tan} = \lim_{h\to 0} \dfrac{f(x_0+h) - f(x_0)}{h}$

$$\begin{aligned} m_{tan} &= \lim_{h\to 0} \frac{\sqrt{x_0+h} - \sqrt{x_0}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x_0+h} - \sqrt{x_0}}{h} \frac{\sqrt{x_0+h} + \sqrt{x_0}}{\sqrt{x_0+h} + \sqrt{x_0}} \\ &= \lim_{h\to 0} \frac{x_0+h-x_0}{h\left(\sqrt{x_0+h} + \sqrt{x_0}\right)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x_0+h} + \sqrt{x_0}} \\ &= \frac{1}{2\sqrt{x_0}} \\ &= \frac{1}{2\sqrt{1}} \\ &= \boxed{\frac{1}{2}.} \end{aligned}$$

Now using the point-slope form at the point $(1,1)$ we have

$$\begin{aligned} y - 1 &= \frac{1}{2}(x-1) \\ y &= \frac{1}{2}x - \frac{1}{2} + 1 \\ &\boxed{y = \frac{1}{2}x + \frac{1}{2}.} \end{aligned}$$

Q2 Find the equations of the tangent and normal lines to the curve $y = \sqrt{x+1}$ at the point $(3,2)$ and sketch the curve and the lines on the same graph.

Solution:

Compute $m_{tan} = \lim_{x\to 3} \frac{f(x)-f(3)}{x-3}$.

$$\begin{aligned} m_{tan} &= \lim_{x\to 3} \frac{f(x)-f(3)}{x-3} \\ &= \lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} \\ &= \lim_{x\to 3} \frac{(\sqrt{x+1}-2)(\sqrt{x+1}+2)}{(x-3)(\sqrt{x+1}+2)} \\ &= \lim_{x\to 3} \frac{(\sqrt{x+1})^2 - 2^2}{(x-3)(\sqrt{x+1}+2)} \\ &= \lim_{x\to 3} \frac{x+1-4}{(x-3)(\sqrt{x+1}+2)} \\ &= \lim_{x\to 3} \frac{x-3}{(x-3)(\sqrt{x+1}+2)} \\ &= \lim_{x\to 3} \frac{1}{\sqrt{x+1}+2} = \frac{1}{\sqrt{3+1}+2} = \frac{1}{2+2} = \boxed{\frac{1}{4}}. \end{aligned}$$

So the required tangent line has slope $\frac{1}{4}$ and passes through the point $(3,2)$, so has the equation:

$$y - 2 = \frac{1}{4}(x-3), \quad 4y - 8 = x - 3 \Longrightarrow \boxed{x - 4y + 5 = 0}.$$

Then the required normal line has slope $-\frac{1}{\left(\frac{1}{4}\right)} = -4$ and passes through the point $(3,2)$, so has the equation:

$$y - 2 = -4(x-3) = -4x + 12 \Longrightarrow \boxed{y + 4x - 14 = 0}.$$

Q3 Let $f(x) = \dfrac{3}{1+2x}$.

1. Find the **slope of the secant line** to the curve $y = f(x)$ passing through points where $x = 1$ and $x = 4$.

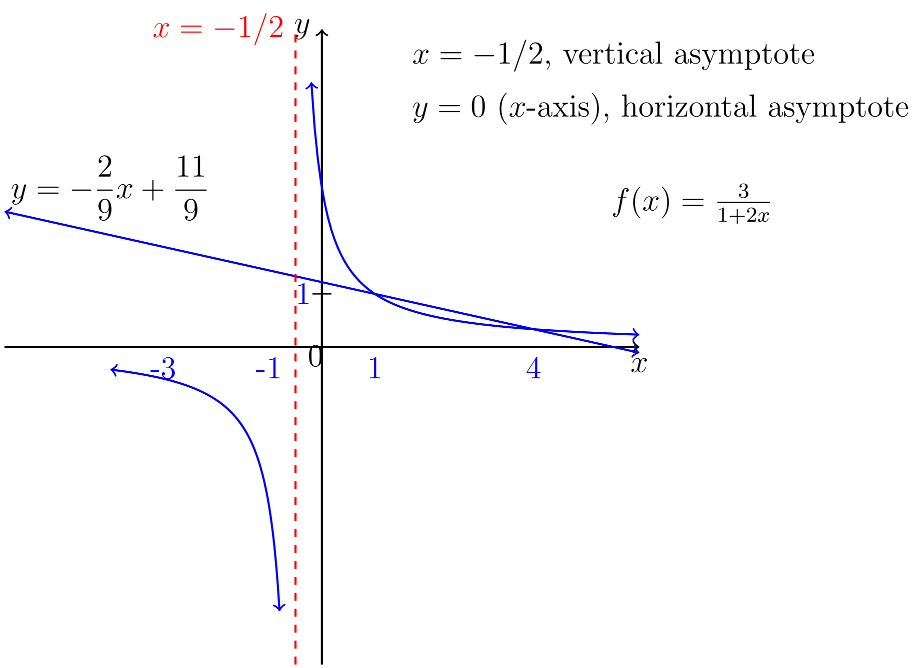
2. Give the **equation of this secant line**.

Solution:

1. $m_{\text{sec}} = \dfrac{f(4) - f(1)}{4-1} = \dfrac{\frac{3}{9} - 1}{3} = -\dfrac{2}{9}$

2. $y - f(1) = m_{\text{sec}}\,(x-1)$

$y - 1 = -\dfrac{2}{9}(x-1) \Longrightarrow \boxed{y = -\dfrac{2}{9}x + \dfrac{11}{9}}$



Q4 Let $f(x) = -2x^2 + 4x + 3$.

1. Using definition, find the **slope of the tangent line** to the curve $y = f(x)$ at $x = 2$.

2. Give the **equation of the tangent line**.

3. Sketch the graph of this tangent line on the graph of $y = f(x)$ at $x = 2$.

Solution:

1) $f(2) = -2(2)^2 + 4(2) + 3 = 3$

$$\begin{aligned}\frac{f(2+h) - f(2)}{h} &= \frac{-2(2+h)^2 + 4(2+h) + 3 - 3}{h}\\ &= \frac{-8 - 8h + 2h^2 + 8 + 4h}{h}\\ &= \frac{-4h + h^2}{h} = \frac{h(-4+h)}{h} = -4 + h\end{aligned}$$

$m_{\tan} = \lim_{h\to 0} \dfrac{f(2+h) - f(2)}{h} = \lim_{h\to 0}(-4+h) = -4$

2) $y - f(2) = m_{\tan}\,(x-2)$, $y - 3 = (-4)(x-2) \Longrightarrow \boxed{y = -4x + 11}$

3.)

Q5 The height of a ball thrown vertically upwards with velocity 100 m/s is given by $f(t) = 100t - t^2$. Find the average velocity in the time intervals

1. $[1, 2]$

2. $[1, 1.01]$

3. Estimate the instantaneous velocity after one second.

Solution:

The average velocity of a particle whose height (or distance) is given by $f(t)$, over a time interval $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}.$$

We have $f(t) = 100t - t^2$ and so

(1) average velocity $= \frac{f(2)-f(1)}{2-1} = \frac{100(2)-2^2-(100(1)-1^2)}{1} = \frac{196-99}{1} = 97.$

(2) $\frac{f(1.01)-f(1)}{1.01-1} = \frac{100(1.01)-(1.01)^2-(100(1)-1^2)}{.01} = 97.99.$

(3)

$$\begin{aligned} \text{instantaneous velocity} &= \lim_{t\to 1} \frac{f(t)-f(1)}{t-1} \\ &= \lim_{t\to 1} \frac{-t^2+100t-99}{t-1} \\ &= \lim_{t\to 1} -(t-99) = 98 \end{aligned}$$

Solution:

Q6 Let $f(x) = x^2 + 2x + 3$, defined for any real number x.
Compute and interpret geometrically, using a graph of the function $y = f(x)$, the following quantities:

1. $\dfrac{f(x)-f(2)}{x-2}$
2. $\lim_{x\to 2}\left(\dfrac{f(x)-f(2)}{x-2}\right)$
3. $\dfrac{f(3+h)-f(3)}{h}$
4. $\lim_{h\to 0}\left(\dfrac{f(3+h)-f(3)}{h}\right)$

Solution:

1. $\dfrac{f(x)-f(2)}{x-2}$

   This is the slope of the secant line joining the points of the graph $(2, f(2))$ and $(x, f(x))$.
   We have $f(2) = 2^2 + 2(2) + 3 = 4 + 4 + 3 = 11$, so we get:

   $$\boxed{\frac{f(x)-f(2)}{x-2} = \frac{x^2+2x+3-11}{x-2} = \frac{(x-2)(x+4)}{x-2} = x+4}$$

2. $\lim_{x\to 2}\left(\dfrac{f(x)-f(2)}{x-2}\right)$

   This is the limit as x goes to 2 of the slope of the secant line joining the points of the graph $(2, f(2))$ and so is the slope of the tangent line at $(2, f(2))$.

   $$\boxed{\lim_{x\to 2}\left(\frac{f(x)-f(2)}{x-2}\right) = \lim_{x\to 2}(x+4) = 2+4 = 6}$$

   So the tangent line to the graph at $(2, 11)$ has slope 6.

3. $\dfrac{f(3+h)-f(3)}{h}$

   This is the slope of the secant line joining the points of the graph $(3, f(3))$ and $(3+h, f(3+h))$.
   We have $f(3) = 3^2 + 2(3) + 3 = 9 + 6 + 3 = 18$.
   Also we have $f(3+h) = (3+h)^2 + 2(3+h) + 3 = 9 + 6h + h^2 + 6 + 2h + 3 = h^2 + 8h + 18$.
   So the slope of the secant line is:

   $$\boxed{\frac{f(3+h)-f(3)}{h} = \frac{h^2+8h+18-18}{h} = \frac{h^2+8h}{h} = h+8}$$

4. $\lim_{h\to 0}\left(\dfrac{f(3+h)-f(3)}{h}\right)$

   This is the limit as $3+h$ goes to 3 of the slope of the secant line joining the points of the graph $(3, f(3))$ and $(3+h, f(3+h))$, so is the slope of the tangent line at $(3, f(3))$.
   We have:

   $$\boxed{\lim_{h\to 0}\left(\frac{f(3+h)-f(3)}{h}\right) = \lim_{h\to 0}(h+8) = 0+8 = 8}$$

   So the tangent line to the graph at $(3, 11)$ has slope 8.

Q7 The height h of a ball after it is thrown vertically upward is given in meters by the function

$$h = \frac{5}{6}(100t - t^2).$$

Find its average vertical velocity over the following time intervals:

1. $[2, 3]$
2. $[2, 2.1]$
3. $[2, 2.01]$
4. $[1.9, 2.1]$
5. Also determine its instantaneous vertical velocity at $t = 2$.

Solution:

1. $[2, 3]$

   The average vertical velocity over the time interval $[2, 3]$ is:

$$\frac{h(3) - h(2)}{3 - 2} = \frac{h(3) - h(2)}{1} = h(3) - h(2) = \frac{5}{6}(300 - 9) - \frac{5}{6}(200 - 4)$$

$$= \frac{5}{6}(291 - 196) = \boxed{\frac{475}{6}}.$$

2. $[2, 2.1]$

   The average vertical velocity over the time interval $[2, 2.1]$ is:

$$\frac{h(2.1) - h(2)}{2.1 - 2} = \frac{h(2.1) - h(2)}{0.1} = 10(h(2.1) - h(2)) = \boxed{\frac{959}{12}}.$$

3. $[2, 2.01]$

   The average vertical velocity over the time interval $[2, 2.01]$ is:

$$\frac{h(2.01) - h(2)}{2.01 - 2} = \frac{h(2.01) - h(2)}{0.01} = 100(h(2.01) - h(2)) = \boxed{\frac{959.9}{12}}.$$

4. $[1.9, 2.1]$

   The average vertical velocity over the time interval $[1.9, 2.1]$ is:

$$\frac{h(2.1) - h(1.9)}{2.1 - 1.9} = \frac{h(2.1) - h(1.9)}{0.2} = \boxed{80}.$$

5. The instantaneous vertical velocity at $t = 2$ is:

$$\begin{aligned}
\lim_{t\to 2} \frac{h(t) - h(2)}{t - 2} &= \lim_{t\to 2} \frac{\frac{5}{6}(100t - t^2) - \frac{5}{6}(196)}{t - 2} \\
&= \lim_{t\to 2} \frac{\frac{5}{6}(-t^2 + 100t - 196)}{t - 2} \\
&= \lim_{t\to 2} \frac{\frac{5}{6}(t - 2)(-t + 98)}{t - 2} \\
&= \lim_{t\to 2} \frac{5}{6}(98 - t) = \frac{5}{6}(98 - 2) = \boxed{80}.
\end{aligned}$$

Q8 Using the definition of the derivative find the derivative of

$$f(x) = \sin x$$

Solution:

We need to use the relation

$$\sin x - \sin y = 2\cos\left(\frac{x + y}{2}\right) \cdot \sin\left(\frac{x - y}{2}\right)$$

Now using the definition of the derivative we have:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sin(x + h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{2\cos\left(\frac{2x+h}{2}\right) \cdot \sin\left(\frac{h}{2}\right)}{h} \\
&= \lim_{h\to 0} \cos\left(\frac{2x + h}{2}\right) \cdot \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \\
&= \boxed{\cos x.}
\end{aligned}$$

Q9 Find the equation of the tangent line to $y = x^3 - 2x^2 + 2$ at $x = 1$.

Solution:

We first begin by evaluating the slope of the tangent to the given curve at the given point. This is found by taking the derivative:

$$y' = 3x^2 - 4x$$

and substituting $x = 1$ which yields $y'(1) = -1$. Now the equation of the tangent is given by the point-slope formula:

$$y - 1 = (-1)(x-1) \quad \text{or} \quad \boxed{y = -x + 2}$$

Q10 Suppose that $f(2) = 3$, $f'(2) = -1$, $g(2) = 5$ and $g'(2) = -2$. Find the derivative of the product $f(x)g(x)$ at $x = 2$.

Solution:

According to the product rule, we have:

$$(fg)'(2) = f'(2)g(2) + f(2)g'(2) = (-1)\cdot 5 + 3 \cdot (-2) = -11$$

Q11 Differentiate the function $f(x) = \sqrt{x} - \dfrac{1}{\sqrt{x}}$

Solution:

$$\boxed{f(x) = x^{\frac{1}{2}} - x^{\frac{-1}{2}} \Longrightarrow f'(x) = \frac{1}{2}x^{\frac{-1}{2}} + \frac{1}{2}x^{\frac{-3}{2}} = \frac{1}{2\sqrt{x}} + \frac{1}{2x\sqrt{x}}}$$

Q12 Differentiate the function $y = f(x) = \dfrac{(x^2 + 4x + 3)}{\sqrt{x}}$

Solution:

$$y = f(x) = x^{\frac{3}{2}} + 4x^{\frac{1}{2}} + 3x^{\frac{-1}{2}}$$

$$\boxed{y' = \frac{3}{2}x^{\frac{1}{2}} + 4\left(\frac{1}{2}\right)x^{\frac{-1}{2}} + 3\left(\frac{-1}{2}\right)x^{\frac{-3}{2}} = \frac{3}{2}\sqrt{x} + \frac{2}{\sqrt{x}} - \frac{3}{2x\sqrt{x}}}$$

Q13 Find the *nth* derivative of the function $f(x) = \dfrac{1}{x}$ by calculating the first few derivatives and observing the pattern that occurs

Solution:

$f(x) = x^{-1} \to f'(x) = (-1)x^{-2} \to f''(x) = (-1)(-2)x^{-3} \to$
$f'''(x) = (-1)(-2)(-3)x^{-4} \to ..... \to$

$$\boxed{f^k(x) = (-1)(-2)(-3).....(-k)x^{-(k+1)} = (-1)^k k! x^{-(k+1)} = \frac{(-1)^k k!}{x^{k+1}}.}$$

Q14 The equation of the motion of a particle is $f(t) = t^3 - 3t$, where f is in meters and t is in seconds. Find

1. Find the velocity and acceleration as functions of t.
2. The acceleration after 2s.
3. The acceleration when the velocity is 0.

Solution:

1. $v(t) = f'(t) = 3t^2 - 3, \;\; a(t) = v'(t) = 6t$
2. $a(2) = 12\dfrac{m}{s^2}$

3. $v(t) = 3t^2 - 3 = 0 \to 3t^2 = 3 \to t^2 = 1 \to t = -1, 1$ (disregard -1)

$a(1) = 6\frac{m}{s^2}$

Q15 Find $\frac{dy}{dx}$ where

1. $y = \left(-3x^2 + \sqrt{x} - \frac{1}{x}\right)^{2003}$

2. $y = \sqrt{\frac{x+1}{x-1}}$

3. $y = \tan(4x - 1)$

4. $y = \ln(2^x + e^{-x})$

Solution:

1. $u = -3x^2 + \sqrt{x} - \frac{1}{x}$, $y = u^{2003}$

$$\frac{dy}{du} = 2003u^{2002}, \quad \frac{du}{dx} = -6x + \frac{1}{2}x^{-1/2} + x^{-2}$$
$$\frac{dy}{dx} = 2003\left(-3x^2 + \sqrt{x} - \frac{1}{x}\right)^{2002}\left(-6x + \frac{1}{2}x^{-1/2} + x^{-2}\right)$$

2. $u = \frac{x+1}{x-1}$, $y = \sqrt{u}$

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2}, \quad \frac{du}{dx} = \frac{(1)(x-1) - (x+1)(1)}{(x-1)^2} = -\frac{2}{(x-1)^2}$$
$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{x+1}{x-1}\right)\left(-\frac{2}{(x-1)^2}\right) = -\frac{x+1}{(x-1)^3}$$

3. $u = 4x - 1$, $y = \tan(u)$

$$\frac{du}{dx} = 4, \quad \frac{dy}{du} = \sec^2(u), \quad \frac{dy}{dx} = 3\sec^2(4x - 1)$$

4. $u = 2^x + e^{-x}$, $y = \ln u$

$$\frac{du}{dx} = 2^x \ln 2 - e^{-x}, \quad \frac{dy}{du} = \frac{1}{u}$$
$$\frac{dy}{dx} = \frac{1}{2^x + e^{-x}}\left(2^x \ln 2 - e^{-x}\right) = \frac{2^x \ln 2 - e^{-x}}{2^x + e^{-x}}$$

Q16 Find $\frac{dy}{dx}$ where $y = \ln\left(\cos\left(e^{x^2 - \sec x}\right)\right)$.

Solution:

$w = x^2 - \sec x$, $v = e^w$, $u = \cos(v)$, $y = \ln(u)$

$$\frac{dw}{dx} = 2x - \sec x \tan x, \quad \frac{dv}{dw} = e^w, \quad \frac{du}{dv} = -\sin v, \quad \frac{dy}{du} = \frac{1}{u}$$
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dw}\frac{dw}{dx}$$
$$\frac{dy}{dx} = \frac{1}{2x - \sec x \tan x}\left(-\sin e^{x^2 - \sec x}\right) e^{x^2 - \sec x}(2x - \sec x \tan x)$$
$$= \frac{-e^{x^2 - \sec x} \sin e^{x^2 - \sec x}(2x - \sec x \tan x)}{2x - \sec x \tan x}$$

Q17 Find $\frac{dy}{dx}$ where $y = \cos(\ln(\sec x + 1))$.

Solution:

$v = \sec x + 1$, $u = \ln v$, $y = \cos u$

$$\frac{dv}{dx} = \sec x \tan x, \quad \frac{du}{dv} = \frac{1}{v}, \quad \frac{dy}{du} = -\sin u$$
$$\frac{dy}{dx} = -\sin(\ln(\sec x + 1))\left(\frac{1}{\sec x + 1}\right)(\sec x \tan x)$$
$$= -\left(\frac{\sec x \tan x}{\sec x + 1}\right)\sin(\ln(\sec x + 1))$$

Q18 Differentiate the function: $f(x) = \tan\left(e^{1/x^2}\right)$.

Solution:

We apply the chain rule twice to get:

$$\left(\tan\left(e^{1/x^2}\right)\right)' = \frac{1}{\cos^2\left(e^{1/x^2}\right)}\left(e^{1/x^2}\right)'$$
$$= \frac{1}{\cos^2\left(e^{1/x^2}\right)}e^{1/x^2}\left(\frac{1}{x^2}\right)'.$$
$$= -\frac{1}{\cos^2\left(e^{1/x^2}\right)}e^{1/x^2}\frac{2}{x^3}$$

Q19 Two functions f and g are such that $f'(5) = 2$, $g(3) = 5$ and $g'(3) = 7$. If h is defined by $h(x) = f(g(x))$, then compute $h'(3)$.

Solution:

We will apply the chain rule according to which:

$$h'(x) = (f(g(x)))' = f'(g(x))g'(x)$$

If we substitute $x = 3$, we get:

$$h'(3) = f'(g(3))g'(3) = f'(5)g'(3) = 2 \cdot 7 = 14$$

Q20 Compute the derivative of the function:

$$f(x) = \frac{x^2e^x + 1}{xe^x + 2}$$

Solution:

We are going to be using the quotient rule and the product rule in order to differentiate this function:

$$\left(\frac{x^2e^x + 1}{xe^x + 2}\right)' = \frac{(x^2e^x + 1)'(xe^x + 2) - (x^2e^x + 1)(xe^x + 2)'}{(xe^x + 2)^2} =$$
$$= \frac{((x^2)'e^x + x^2(e^x)' + 0)(xe^x + 2) - (x^2e^x + 1)((x)'e^x + x(e^x)' + 0)}{(xe^x + 2)^2} =$$
$$= \frac{(2xe^x + x^2e^x)(xe^x + 2) - (x^2e^x + 1)(e^x + xe^x)}{(xe^x + 2)^2}$$

Q21 A curve is given by $y^3 + 4xy - x^3 + x^2y = 5$. Find the equation of the tangent to the curve at the point $(1, 1)$.

Solution:

Q22 Find the equation of the tangent to the the curve $yx^2 + y^3x + y = 7$ at the point $(2, 1)$.

Solution:

We first differentiate both sides considering y a function of x:
$(yx^2 + y^3x + y)' = (7)' \implies y'x^2 + 2xy + 3y^2y'x + y^3 + y' = 0$
and we, then, solve for y':

$$y' = \frac{-y^3 - 2xy}{x^2 + 3y^2x + 1}$$

We, now, substitute $x = 2$ and $y = 1$:

$$y' = \frac{-5}{11}$$

We finally use the point-slope formula:

$$y - 1 = -\frac{5}{11}(x - 2)$$

Q23 Let functions $f(x)$ and $g(x)$ obey the following properties:

$$f(3) = 4, \quad f'(3) = -2, \quad g(3) = -5, \quad g'(3) = -1.$$

1. Find the equation of the tangent line to the curve $y = f(x)g(x)$ at $x = 3$.
2. Let $p(x) = f^3(x) + g^3(x)$. Find $p'(3)$.

Solution:

Q24 Find $\frac{dy}{dx}$ by implicit differentiation if $x^2 - 2xy + y^3 = 5x$

Solution:

$$(x^2 - 2xy + y^3 - 5x)' = (x^2)' - (2xy)' + (y^3)' - (5x)' = 0$$

$$2x - ((2x)'y + 2xy') + 3y^2y' - 5 = 0$$

$$2x - (2y + 2xy') + 3y^2y' - 5 = 2x - 2y - 2xy' + 3y^2y' - 5 = 0$$

$$(3y^2 - 2x)y' = 2y - 2x + 5$$

$$\boxed{y' = \frac{2y - 2x + 5}{3y^2 - 2x}}$$

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Q25 Let $y = y(x)$ be implicitly defined by $x = y + \sin(y)$.
Find $y(0)$, $y'(0)$ and $y''(0)$.

Solution:

Q26 Verify that the function $f(x) = x^3 + x - 1$ satisfies the hypothesis of the Mean Value Theorem on the interval $[0, 2]$. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

Solution:

Q27