Instructions: Show all of your work, and clearly indicate your answers.

Q1 Let $f(x) = \sqrt{3x-2}$. Compute f' at $x = 2$ by the definition (that is, use the four step process)

Solution:

Q2 Show that the function $f(x) = x|x|$ is differentiable at every real number.

Solution:

Q3 Let $f(x) = \dfrac{1}{x+1}$.
(a) Write the derivative $f'(3)$ as a limit of an appropriate difference quotient.
(b) Evaluate this limit to find $f'(3)$.

Solution:

Q4 Let f be a function on $\mathbb{R}$ which is continuous at 0 with $f(0) = 2$. Show that if g is the function defined by $g(x) := xf(x)$, then g is differentiable at 0. Moreover compute $g'(0)$.

Solution:

Q5 Determine whether or not $f'(0)$ exists.

$$f(x) = \begin{cases} x\sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Solution:

Q6 Determine whether or not $f'(0)$ exists

$$f(x) = \begin{cases} x^2\sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Solution:

Q7 Find the derivatives of the following functions at $x = 3$ given $f(3) = 6,\ f'(3) = 0,\ f(1) = 1,\ f'(1) = 5,\ g(3) = 1,\ g'(3) = 2,$ $g(4) = 2g'(4) = 1,\ h(3) = 4,$ and $h'(3) = 2$

1. $f(g(x))h(x)$
2. $\frac{f(x)}{g(h(x))}$

Solution:

Q8 Find y' if

$$y = \tan(\cos(\cos(x^2)))$$

Find $y^{(1123)}$ if

$$y = \sin^2 x$$

Solution:

Q9 Given the following functions find y'

Solution:

Q10 The leaning Tower of Pisa is 180 feet high. A ball dropped from the top will fall $16t^2$ feet in t seconds.

1. Hence the ball will be ____________ feet above the ground t seconds after release.
2. After one second, the ball will be ____________ feet above the ground.
3. After two seconds, the ball will be ____________ feet above the ground.
4. During the first second, the ball will drops ____________ feet. During the next second. it drops ____________ feet.

5. At time t the ball is $180 - 16t^2$ feet above the ground. One second later, at a time $t+1$, it is ____________ feet above the ground.

6. During the one second interval between time t and $t+1$, the ball drops ____________

Solution:

Q11 Find an equation of the tangent line to the curve with parametric equations $x = t\sin t; y = t\cos t$ at the point $(0, -\pi)$

Solution:

Q12 Find $\frac{dy}{dx}$ by implicit differentiation $\cos(x-y) = xe^x$

Solution:

Q13 Show that the sum of the x and y intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to c.

Solution:

Q14 Find y' if $x^y = y^x$

Solution:

Q15 If a snowball melts so that its surface area decreases at a rate of $1\frac{cm^2}{\text{min}}$, find the rate at which the diameter decreases when the diameter is 10 cm.

Solution:

Q16 The altitude of a triangle is increasing at a rate of $1\frac{cm}{\text{min}}$ while the area of the triangle is increasing at a rate of $2\frac{cm^2}{\text{min}}$. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2

Solution:

Q17 Show how the Mean Value Theorem applies to $f(x) = x^3 - 6x + 1$ on $[0, 3]$.

Solution:

Q18 Show there does not exist a differentiable function on $[1, 5]$ with $f(1) = -3$ and $f(5) = 9$ with $f'(x) \leq 2$ for all x.

Solution:

Q19 Show how Rolles theorem applies to $f(x) = x^2 - 5x$ on $[1, 4]$

Solution:

Q20 $f(0) = 1$, $f'(x)$ exists for all values of x and $f'(x) \leq 4$ for all x, how large can $f(2)$ possibly be?.

Solution:

Q21 Find the absolute maximum and absolute minimum values of $f\,(x) = x^3 - 6x^2 + 9x + 2$ on $[-1,\ 4]$.

Solution:

Q22 The pressure P of a certain gas is related to its volume V according to the equation $P = \frac{2}{V}$. Find the rate of change of pressure with respect to its volume.

Solution:

Q23 The strength S of a certain beam is related to its thickness t by the equation $S = 2t^2$. Find the rate at which the strength of the beam increases with respect to its thickness.

Solution:

Q24 Find the rate of change of the area of a circle with respect to its radius.

Solution:

The area A of a circle of radius r is $A = \pi r^2$. The required rate of change is

$$\frac{dA}{dr} = 2\pi r$$

.

Q25 Find the rate of change of the volume V of a cube with respect to its side s when its side is $6cm$.

Solution:

Q26 Determine whether the function $f(x) = \frac{1}{x+1}$ is differentiable on the closed interval $[-2, 0]$.

Solution:

Q27 The volume of a sphere of radius r is decreasing at the rate of $6cm^3/hr$. At what rate is its surface area decreasing when its radius is 40 cm?

Solution:

Q28 If $y = A\sin(\ln x) + B\cos(\ln x)$, where A and B are constants, show that

$$x^2y'' + xy' + y = 0.$$

Solution:

Q29 Find the first and second derivative of

$$f(x) = \tan(\ln x^2).$$

Solution:

Q30 Let $f(x) = \frac{x^3+1}{x}$.

1. Find $f'(x)$

2. Find all the points $P(a, f(a))$ where the tangent line to the curve $y = f(x)$ is horizontal, **and** write down the equation of the tangent line at such points.

Solution:

Q32 If $s = \dfrac{2t+5}{t^2-1}$ represents the displacement of a moving particle at time t, what is the velocity v at $t = 0$?

Solution:

Q33 Let $y = u^7 + 7u$; $u = 3x + 71$. Find $\frac{dy}{dx}$.

Solution:

By chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = (7u^6 + 7)\cdot 3 = 21u^6 + 21 = 21(3x+71)^6 + 21.$$

Q34 Use the Mean Value Theorem to prove that $|\sin b - \sin a| \le |b - a|$.

Solution:

Q35 Let $f(x) = x^3 - 3x + 1$.

1. $\lim_{x\to\infty} f(x) = ?$ $\qquad \lim_{x\to-\infty} f(x) = ?$

2. Are there any absolute maximum or minimum points? Where are they if they exist?

3. Find the local maximum point(s) and local minimum point(s), respectively.

4. Find the local maximum point(s) and local minimum point(s), respectively.

5. Find the point(s) of inflection.

Solution:

Q36 Let $y = \frac{\sin x}{\cos x}$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Solution:

Q37 The graph of the equation

$$xy + 4 = x + y^2$$

is a curve that crosses the y-axis at two points, P and Q. Find the point where the tangent lines to the curve at P and at Q cross.

Solution:

Q38 Find an equation for the tangent line to the curve

$$x^2 + xy - y^2 = 1$$

at the point $P_0(2, 3)$.

Solution:

Q39 A ladder 26 ft long leans against a vertical wall. The foot of the ladder is being drawn away from the wall at a rate of 4 ft/sec. How fast is the top of the ladder sliding down the wall at the instant when the foot of the ladder is 10 ft from the wall?

Solution:

Q40 A point moves along the curve $y^2 = x^3$ in such a way that its distance from the origin increases at a constant rate of two units per second. Find dx/dt at the point $(x, y) = (2, 2\sqrt{2})$.

Solution:

Q41 Compute $\frac{dy}{dx}$ for the relation

$$e^{xy} + y^2 = x.$$

Show your work.

Solution:

$$\begin{aligned}
\frac{d}{dx}(e^{xy} + y^2) &= \frac{d}{dx}(x), \\
e^{xy} \cdot \frac{d}{dx}(xy) + 2y\,\frac{dy}{dx} &= 1, \\
e^{xy}\left(y + x\,\frac{dy}{dx}\right) + 2y\,\frac{dy}{dx} &= 1, \\
xe^{xy}\frac{dy}{dx} + 2y\frac{dy}{dx} &= 1 - ye^{xy}, \\
(xe^{xy} + 2y)\frac{dy}{dx} &= 1 - ye^{xy}, \\
\frac{dy}{dx} &= \frac{1 - ye^{xy}}{xe^{xy} + 2y}.
\end{aligned}$$

Q42 Use the definition of the derivative to compute the derivative of $g(x) = -x^2 - 4x + 1$. Show your work.

Solution:

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{[-(x+h)^2 - 4(x+h) + 1] - [-x^2 - 4x + 1]}{h} \\
&= \lim_{h\to 0} \frac{[-(x^2 + 2xh + h^2) - 4x - 4h + 1] - [-x^2 - 4x + 1]}{h} \\
&= \lim_{h\to 0} \frac{-x^2 - 2xh - h^2 - 4x - 4h + 1 + x^2 + 4x - 1}{h} \\
&= \lim_{h\to 0} \frac{-2xh - h^2 - 4h}{h} \\
&= \lim_{h\to 0}(-2x - h - 4) \\
&= -2x - 0 - 4 = -2x - 4.
\end{aligned}$$

Q43 Let f be a function which satisfies $f(2) = 4$, $f(4) = 9$, $f'(2) = 2$, and $f'(4) = -6$. Consider the function $h(x) = f(\sqrt{x})$. Compute $h'(4)$. Show your work.

Solution:

Use the Chain Rule to compute $h'(x) = \frac{d}{dx}f(\sqrt{x}) = f'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$ and $h'(4) = f'(\sqrt{4}) \cdot \frac{1}{2\sqrt{4}} = \frac{f'(2)}{4} = \frac{2}{4} = \frac{1}{2}$.

Q44 Compute $\frac{dy}{dx}$ for the relation

$$\cos(x^2y) - y^3 = x + 2.$$

Show your work.

Solution:

$$\begin{aligned}
\frac{d}{dx}[\cos(x^2y) - y^3] &= \frac{d}{dx}(x+2), \\
-\sin(x^2y) \cdot \frac{d}{dx}(x^2y) - 3y^2\frac{dy}{dx} &= 1, \\
-\sin(x^2y) \cdot \left(2xy + x^2\frac{dy}{dx}\right) - 3y^2\frac{dy}{dx} &= 1, \\
-x^2\sin(x^2y)\frac{dy}{dx} - 3y^2\frac{dy}{dx} &= 1 + 2xy\sin(x^2y), \\
[-x^2\sin(x^2y) - 3y^2]\frac{dy}{dx} &= 1 + 2xy\sin(x^2y), \\
\frac{dy}{dx} &= \frac{1 + 2xy\sin(x^2y)}{-x^2\sin(x^2y) - 3y^2}.
\end{aligned}$$

Q45 Compute the tangent line to the graph of the function $y = \sqrt{x^2 - 5}$ at the point $(x, y) = (3, 2)$. Put your answer in the form $y = mx + b$. Show your work.

Solution:

$$\begin{aligned}
y' &= \frac{1}{2\sqrt{x^2-5}} \cdot 2x \\
&= \frac{x}{\sqrt{x^2-5}}, \\
y'(3) &= \frac{3}{\sqrt{3^2-5}} = \frac{3}{\sqrt{4}} = \frac{3}{2}, \\
y - y_0 &= m(x - x_0), \\
y - 2 &= \frac{3}{2}(x-3), \\
y &= \frac{3}{2} \cdot x - \frac{9}{2} + 2 = \frac{3}{2} \cdot x - \frac{5}{2}.
\end{aligned}$$

Q46
Assume the position of a bicycle traveling along a road is $15t + \cos 6t$ miles, with t in hours.

1. Compute the velocity of the bicycle at time t.
2. Compute the acceleration of the bicycle at time t.

Solution:

1. Compute the velocity of the bicycle at time t.

   The position is $s = 15t + \cos 6t$ and the velocity is

   $$v = s' = 15 - \sin 6t \cdot 6 = 15 - 6\sin 6t$$

   .

2. Compute the acceleration of the bicycle at time t.

   The acceleration $a = v' = 0 - 6\cos 6t \cdot 6 = -36\cos 6t$.

Q47
Consider the function $g(x) = x - \ln x$.

1. What is the domain of $g(x)$?

2. Find all the intervals on which g is increasing. Also find all intervals on which g is decreasing. Find all critical points of g. Show your work.

3. Find all intervals on which g is concave up. Find all intervals on which g is concave down. Show your work.

Solution:

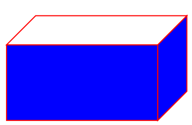
1. The domain of $\ln x$ is $x > 0$, and so $x > 0$ is the domain of $g(x)$ also.

2. Compute $g'(x) = 1 - \frac{1}{x}$ and so $g'(x) = 0$ if $1 - \frac{1}{x} = 0$, $x - 1 = 0$, $x = 1$. So $x = 1$ is the only critical point in the domain $(0, \infty)$. Now check the sign of $g'(x)$ for x in the subintervals $(0, 1)$ and $(1, \infty)$: if $x = \frac{1}{2}$, $g'(\frac{1}{2}) = 1 - \frac{1}{\frac{1}{2}} = 1 - 2 < 0$. Therefore $g'(x) < 0$ for x in $(0, 1)$. $g(x)$ is decreasing there. On the other hand for $x = 2$, $g'(2) = \frac{1}{-}\frac{1}{2} = \frac{1}{2} > 0$, and so $g'(x) > 0$ for x in $(1, \infty)$. So $g(x)$ is increasing there.

3. Compute $g''(x) = \frac{1}{x^2}$. This is always positive. So $g(x)$ is concave up on its domain $(0, \infty)$.

Q48 A box with a square base and open top must have a volume of $32,000$ cm^3. Find the dimensions of the box that minimize the amount of material used.

Solution:



Let x, y be the base and height of the box. Then we have:

1. Volume: $V = x^2 y = 32,000$

2. $y = \dfrac{32,000}{x^2}$

3. The surface area of the box is:

$$S = x^2 + 4xy = x^2 + 4x\left(\frac{32,000}{x^2}\right) = x^2 + 4\left(\frac{32,000}{x}\right)$$

4. Critical points:

$$S' = 2x - 4\left(\frac{32,000}{x^2}\right) = 0$$

$$\left(\frac{128,000}{x^2}\right) = 2x \Longrightarrow x^3 = 64,000 \Longrightarrow x = \sqrt[3]{64,000} = 40$$

5. This is an **absolute minimum** because $S' < 0$ if $x < 40$, and $S' > 0$ if $x > 40$.

6. Therefore the dimensions should be: $40 \times 40 \times 20$

Q49 Show that the rectangle with the largest area inscribed in a circle is a square.

Solution:

Q50 Given f is a continuous function and $f(0) = 0$ use the following table to graph the function f

x	$-\infty$		1		4		6		$+\infty$
$f'(x)$		$+$		$-$		$-$		$+$	
$f''(x)$		$-$		$-$		$+$		$+$	

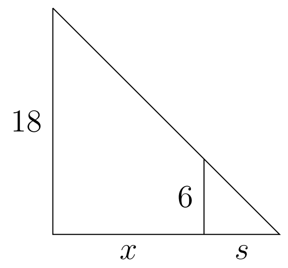
Solution:

Q51

Suppose a 6ft tall person is 12ft away from a 18ft tall lamppost. If the person is moving away from the lamppost at a rate of 2ft/s, at what rate is the length of the shadow changing?

Solution:

Using similar triangles, we have



$$x = 2s$$

so by implicit differentiation,

$$\frac{dx}{dt} + \frac{ds}{dt} = 3\frac{ds}{dt}$$
$$\frac{ds}{dt} = \frac{1}{2}\frac{dx}{dt}$$
$$\boxed{\frac{ds}{dt} = \frac{1}{2}\frac{dx}{dt} = \frac{1}{2}\cdot 2 = 1}$$

Q52 Show that if $x, y,$ and z are positive numbers then

$$\frac{(x^2+1)(y^2+1)(z^2+1)}{xyz} \geq 8$$

Solution:

If we minimize each of $f(x) = \dfrac{x^2+1}{x}$, $g(y) = \dfrac{y^2+1}{y}$, and $h(z) = \dfrac{z^2+1}{z}$ then we have minimized the product. Find the minimum of $f(x) = \dfrac{x^2+1}{x}$.

⋮

⋮

Therefore

$$\frac{(x^2+1)(y^2+1)(z^2+1)}{xyz} \geq 2\cdot 2\cdot 2 = 8.$$

Q53 Show that for a rectangle of given perimeter 4, the one with the maximum area is a square.

Solution:

The perimeter $P = 4 = 2x + 2y$, while the area $A = xy$. We want to maximize A. It is clear that $x = 2 - y$. Therefore

$$A = xy = (2-y)y = 2y - y^2.$$

Let us find the critical points.

1. Compute
$$\frac{dA}{dy} = 2 - 2y, \qquad \frac{d^2A}{dy^2} = -2.$$
So the only critical point is when $2 - 2y = 0$, which is when $y = 1$.

2. Second derivative test: $A''(1) = -2 < 0$ and the second derivative test shows $y = 1$ is a local maximum. Since this is the only critical point, this must be the global maximum.

3. For $y = 1$, then we find $x = 2 - y = 2 - 1 = 1$.

4. So for the maximum area, the two sides $y = 1$ and $x = 1$, and so the rectangle is a **square.**

Q54 What is the largest possible product you can form from two non-negative numbers whose sum is 30?

Solution:

1. Given information: $x \geq 0, y \geq 0$ and $x + y = 30$

2. We want to maximize the product of x and y.

3. $P = xy$

4. Solve for y in terms of x: $y = 30 - x$

5. $P = x(30 - x) = -x^2 + 30x$

6. We know $x \geq 0$ and $y = 30 - x \geq 0$ so $x \leq 30$.

7. We need to maximize $P = -x^2 + 30x$ on $[0, 30]$.

8. Critical points: $P'(x) = -2x + 30 = 0 \Longrightarrow x = 15$.

9. Apply extreme value theorem: $P(0) = 0, P(15) = 224, P(30) = 0$

10. Largest product is 225.

Q55 Suppose the product of x and y is 26 and both x and y are positive. What is the minimum possible sum of x and y?

Solution:

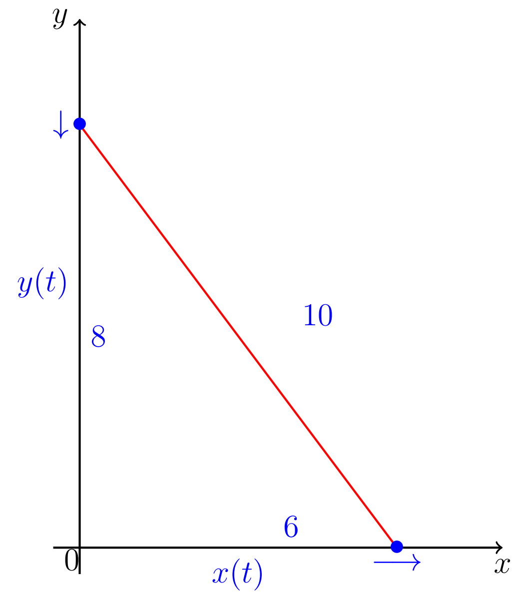
Q56 Let A be the point $(0, 1)$ and let B be the point $(5, 3)$. Find the length of the shortest path that connects points A and B if the path must touch the x-axis. In other words, the path goes from point A to somewhere (say P) on the x-axis, and then to B.

Solution:

minimize $D = \sqrt{1 + x^2} + \sqrt{(5 - x)^2 + 9}$.

Q57 A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 feet/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 feet from the wall?

Solution:



Length of the ladder is $D = 10$. We know $dx/dt = 1 feet/sec$. What is $dy/dt = ?$.