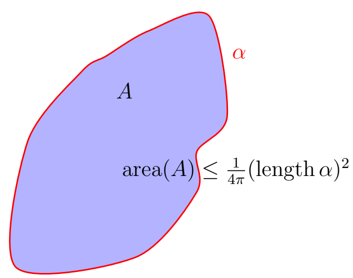
## Isoperimetric Inequality/Surfaces in $\mathbb{R}^3$

Theorem Let $\alpha$ be a simple, closed, regular curve of length $L$. Let $A$ be the area bounded by $\alpha$. Then,

$$A \leq \frac{L^2}{4\pi}$$

and $A = \frac{L^2}{4\pi}$ if and only if the image of $\alpha$ is a circle.




Question Show that the ratio $\frac{A}{L^2}$ is independent of the size of the figure and depends only on the shape of the boundary.

Note If the perimeter of a region is changed by a factor of $\lambda$, the area changes by a factor of $\lambda^2$. Given an arbitrary curve, we can thus rescale,

$$\frac{A'}{L'^2} = \frac{\lambda^2 A}{(\lambda L)^2} = \frac{A}{L^2}$$

Definition [Open Set] We say a set $\mathcal{U} \in \mathbb{R}^2$ is *open* if for each point $(a, b) \in \mathcal{U}$ , $\exists \epsilon > 0$ such that each point $(x, y)$ satisfying

$$((x-a)^2 + (y-b)^2)^{\frac{1}{2}} < \epsilon$$

also lies in $\mathcal{U}$.

Notation Let $\mathbf{r} : \mathcal{U} \to \mathbb{R}^3$. Then $\mathbf{r} = \mathbf{r}(u, v)$. We can write this in component form as,

$$\mathbf{r} = (x(u, v), y(u, v), z(u, v))$$

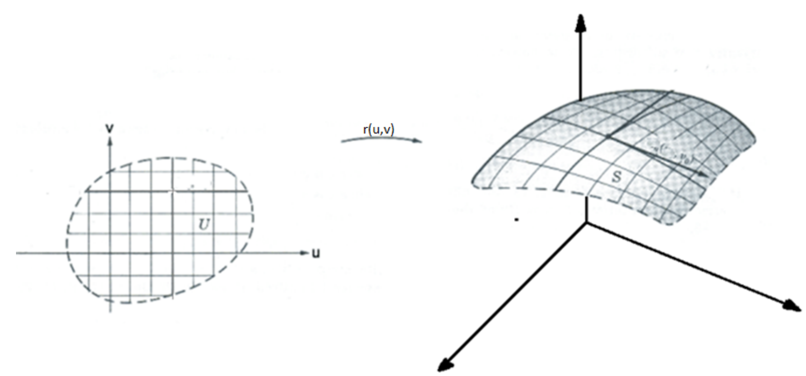
Notation Partial derivatives of a vector-valued function $\mathbf{r}$ with respect to the variable $u$ and $v$ are written as $\mathrm{r}_u$ and $\mathrm{r}_v$, respectively.

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} \qquad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$$

Notation [Coordinate patch] A *coordinate patch* (or simple surface) is smooth, one-to-one mapping $\mathbf{r} : \mathcal{U} \to \mathbb{R}^3$ where $\mathcal{U}$ is open and

$$\mathbf{r}_u \times \mathbf{r}_v \neq 0$$

The condition on the cross product implies that both $\mathbf{r}_u$ and $\mathbf{r}_v$ are not equal to 0 and that they are not parallel.




Definition A coordinate patch

$$r : \mathcal{U} \to \mathbb{R}^3$$

is called **proper** if its inver function

$$r^{-1} : r(\mathcal{U}) \to \mathbb{R}^3$$

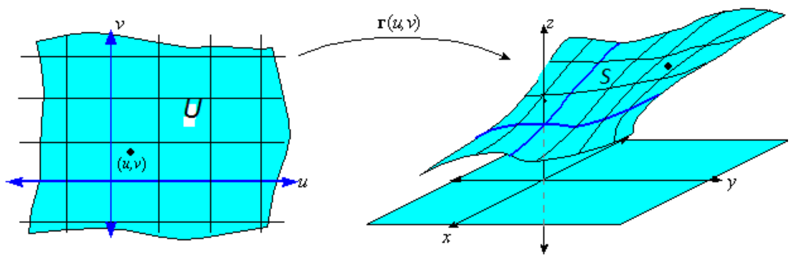
is continuous.

Definition A smooth surface in $\mathbb{R}^3$ is a subset $S \in \mathbb{R}^3$ such that each point has a neighborhood $U \subset S$ and a map $r : V \to \mathbb{R}^3$ from an open set $V \subset \mathbb{R}^2$ such that

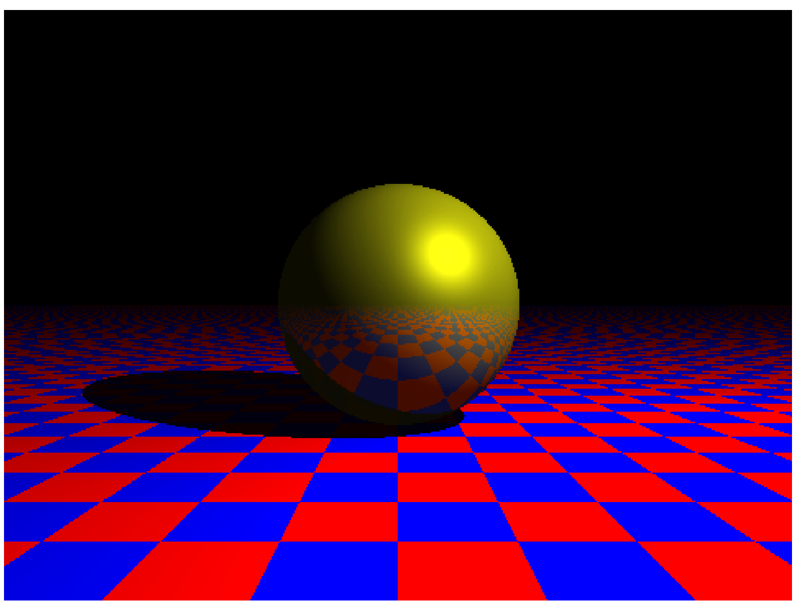
**1.** $r : V \to U$ is a homeomorphism

**2.** $r(u, v) = (x(u, v), y(u, v), z(u, v))$ has derivatives of all orders

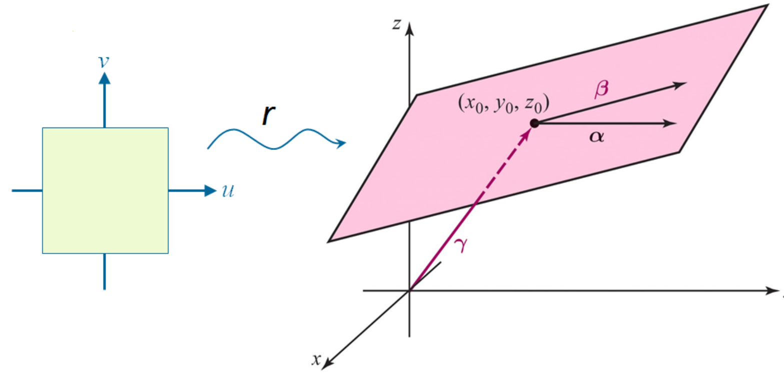
**3.** at each point $r_u = \frac{\partial r}{\partial u}$, and $r_v = \frac{\partial r}{\partial v}$ are linearly independent.




Example A sphere $r(u, v) = (a \sin u \sin v, a \cos u \sin v, a \cos v)$



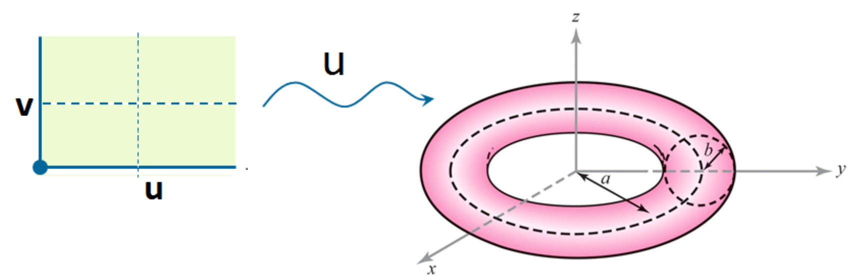
Example [Parametric Equation of the Plane]

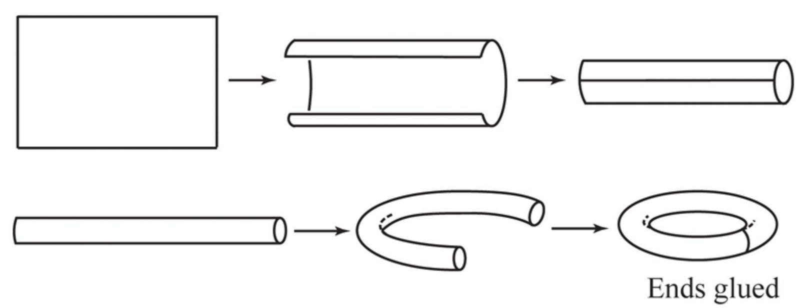


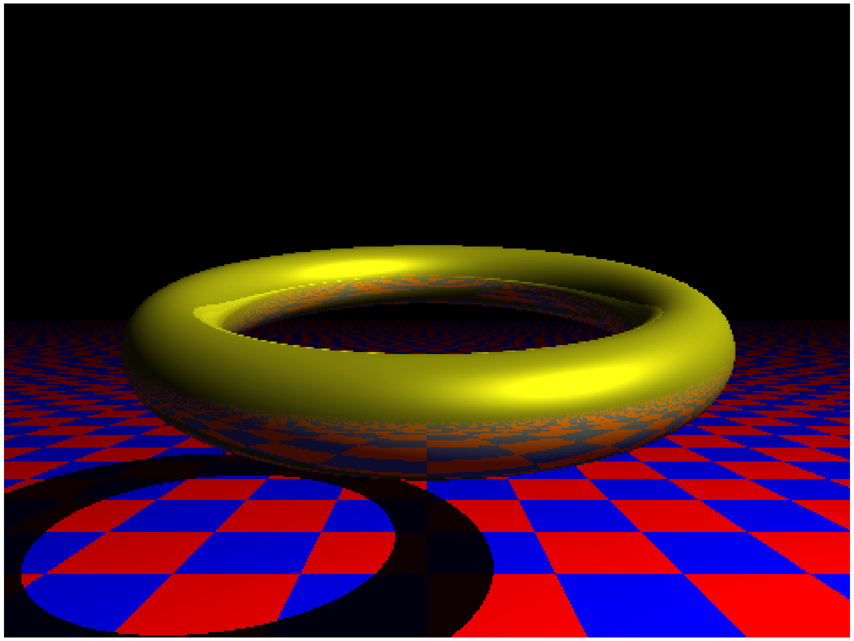

Parametric Equation of the Plane: $r(u, v) = \gamma + u\alpha + v\beta$.

Example [Parametric Equation of Torus]

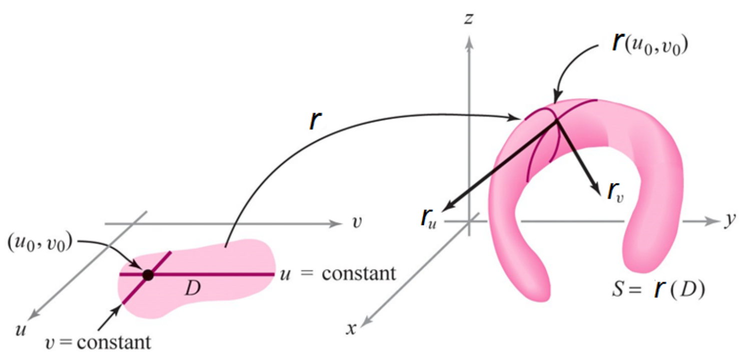
$$r(u, v) = (a + b \cos u)\Big((\cos v)\mathbf{i} + (\sin v)\mathbf{j}\Big) + b \sin u\mathbf{k}$$



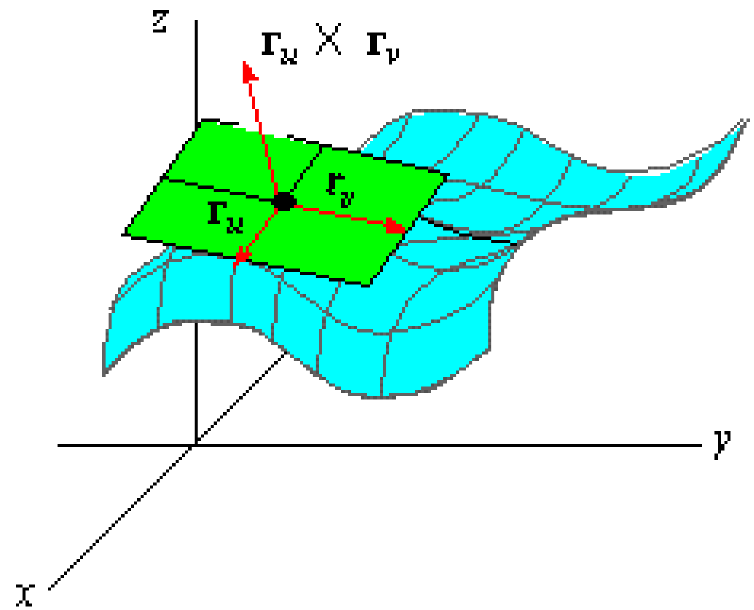





Example [Tangent Vectors to Parametrized Sufaces]




Definition The tangent plane (or tangent space) of a surface at the point $a$ is the vector space spanned by $r_u(a)$, $r_v(a)$ .



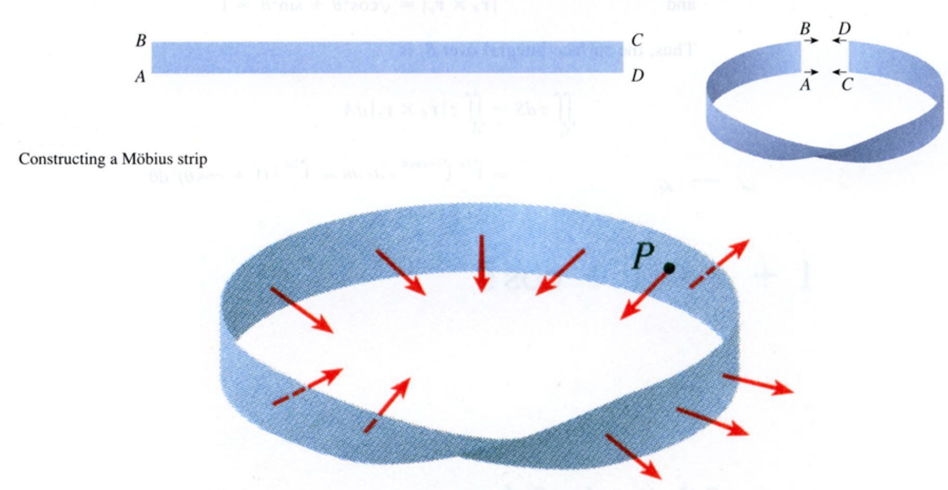

Definition The vectors

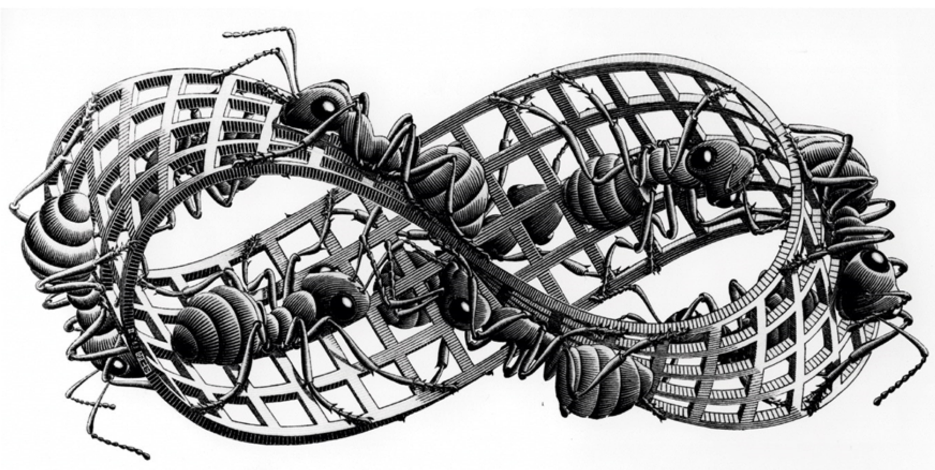
$$n = \pm \frac{r_u \times r_v}{|r_u \times r_v|}$$

are the two unit normals (inward and outward) to the surface at $(u, v)$.

Definition A parametric surface is orientable if $n(u, v)$ varies **continuously** across the surface and **defines only one surface normal at each point on the surface.**

Note Not all surfaces can be oriented. For example, a Möbius strip, which can be formed by twisting a strip of paper one half turn and pasting the two ends together, cannot be oriented).



Definition [Monge patch] A *Monge patch* is a special case of a surface in which the surface is of the form $z = f(x, y)$ for $((x, y) \in \mathcal{U}$ . The surface can be parameterized,

$$\mathbf{r}(u, v) = (u, v, f(u, v))$$

Note A Monge patch is essentially the graph of a function of the form $z = f(x, y)$. $\mathcal{U}$ is the projection of the surface onto the $xy$ plane.

To show that a Monge patch is indeed a surface, we should ensure that the condition of regularity is satisfied,

$$\mathbf{r}_u = \left(1, 0, \frac{\partial f}{\partial u}\right) \qquad \mathbf{r}_v = \left(0, 1, \frac{\partial f}{\partial v}\right)$$
$$\Rightarrow \mathbf{r}_u \times \mathbf{r}_v = \left(-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1\right) \neq 0$$

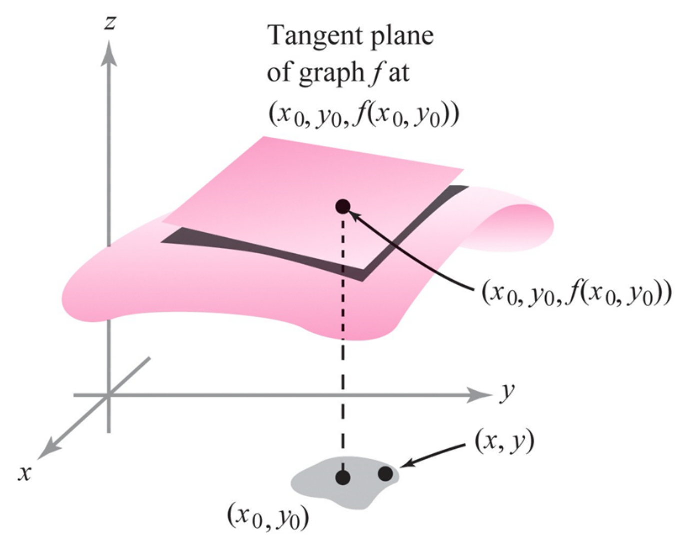
Hence the general Monge patch is a simple surface, as claimed.

Note The vectors $\{\mathbf{r}_u, \mathbf{r}_v, \mathbf{n}\}$ are linearly independent and thus form a basis of $\mathbb{R}^3$, although it is *not* an orthonormal basis.

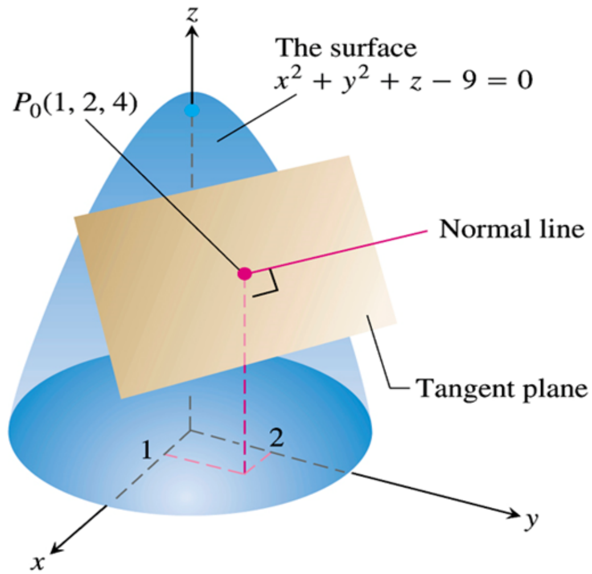
Definition What is the tangent plane to the surface of a function

$$z = f(x, y)$$

at $(x_0, y_0)$ ?




Question Find the tangent plane and the normal line to the surface of the function $z = 9 - x^2 - y^2$ at the point $P_0 = (1, 2, 4)$.




Solution

1. $2(x-1) + 4(y-2) + 1(z-4) = 0$
2. The normal line to surface at $P_0$ is the line through $P_0$ parallel to normal vector.

   $x = 1 + 2t, \quad y = 2 + 4t, \quad z = 4 + t$

Question Find the surface unit normal and the equation of the tangent plane to the cylinder

$$r(u, v) = (3\cos u, 3\sin v, v)$$

at $r(\pi, 2) = (-3, 0, 2)$. Solution

$$n(\pi, 2) = \frac{r_u \times r_v}{|r_u \times r_v|}\Big|_{(\pi,2)} = (\cos u, \sin u, 0)\Big|_{(\pi,2)} = (-1, 0, 0)$$

Thus, the equation of the tangent plane is

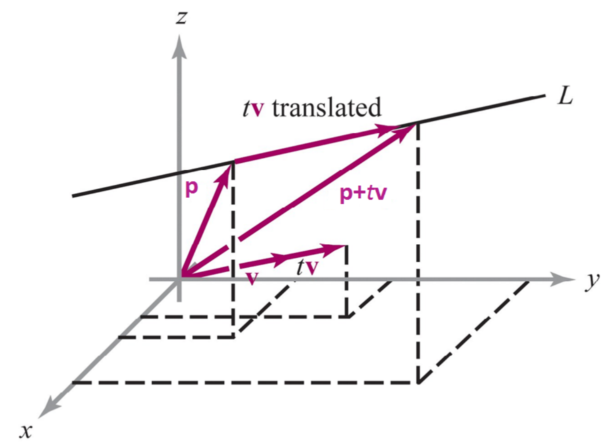
$$-1(x+3) + 0(y-0) + 0(z-0) = 0 \Longrightarrow x = -3$$

Definition [Gradient] If $f : \mathcal{U} \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function, the gradient of $f$ at $(x, y, z)$ is the vector in space given by

$$\nabla f(x, y, z) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}).$$

Definition [Directional Derivatives] If $f : \mathcal{U} \subset \mathbb{R}^3 \to \mathbb{R}$, the directional derivative of $f$ at $p = x_0, y_0, z_0)$ along the vector $\mathbf{v}$ is given by

$$D_{\mathbf{v}} f = \frac{d}{dt} f(p + t\mathbf{v})\Big|_{t=0} = \lim_{t\to 0} \frac{f(p + t\mathbf{v}) - f(p)}{t}.$$




Theorem If $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable, then all directional derivatives exist. The directional derivative of $f$ at $p = (x_0, y_0, z_0)$ in the direction $\mathbf{v}$ is given by

$$D_{\mathbf{v}} f = \nabla f(x, y, z) \cdot \mathbf{v}$$

## Surfaces by level sets

Notation Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function (differentiable) $f = f(x, y, z)$. A *level set* of a function $f$ is a set,

$$M_c = \{(x, y, z) | f(x, y, z) = c\}$$

Theorem Suppose $\nabla f \neq 0$ everywhere on $M_c$. Then $M_c$ is a surface.

Theorem The gradient $\nabla f$ is normal to the surface $M_c$ at each point on the surface.

Question Find the equation of the tangent plane to the circular cone

$$x^2 + y^2 = z^2$$

at the point $(0.6, 0.8, 1)$.

Solution Since the equation of the surface can be written $f(x, y, z) = x^2 + y^2 - z^2 = 0$. The gradient of $f$ is $\nabla f(x, y, z) = (2x, 2y, -2z)$. At the point $(0.6, 0.8, 1)$, the vector $\nabla f(0.6, 0.8, 1) = (1.2, 1.6, 2)$ is normal to the surface. Therefore the equation of the tangent plane is

$$1.2(x - 0.6) + 1.6(y - 0.8) + 2.(z - 1) = 0 \Longrightarrow \boxed{z = 0.6x + 0.8y}$$

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## First Fundamental Theorem

Definition A smooth curve **lying in the surface** is a map

$$t \to \big(u(t), v(t)\big)$$

with derivatives of all orders such that

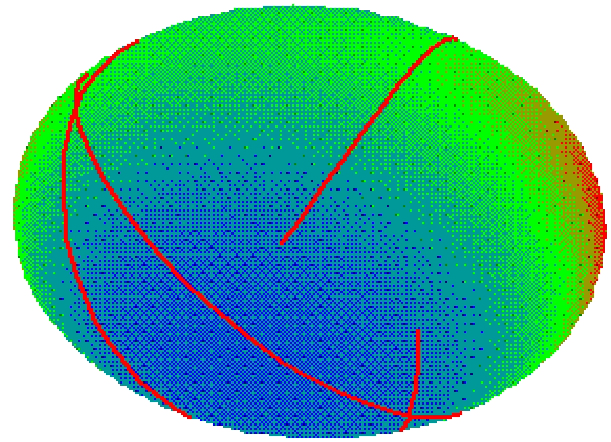
$$\gamma(t) = r\big(u(t), v(t)\big)$$

is a parametrized curve in $\mathbb{R}^3$

Note A parametrized curve means that $u(t), v(t)$ have derivatives of all orders and

$$\gamma' = r_u u' + r_v v' \neq 0.$$

The definition of a surface implies that $r_u$, $r_v$ are **linearly independent**, so this condition is **equivalent to** $(u', v') \neq 0$.



The arc length of such a curve from $t = a$ to $t = b$ is:

$$\begin{aligned} L[\gamma] &= \int_a^b |\gamma'(t)| dt = \int_a^b \sqrt{\gamma'(t) \cdot \gamma'(t)} dt \\ &= \int_a^b \sqrt{(r_u u' + r_v v') \cdot (r_u u' + r_v v')} dt \\ &= \int_a^b \sqrt{E u'^2 + 2F u' v' + G v'^2} dt \end{aligned}$$

where

$$\boxed{E = r_u \cdot r_u,\ F = r_u \cdot r_v,\ G = r_v \cdot r_v}$$

Definition The **first fundamental form** of a surface in $\mathbb{R}^3$ is the expression

$$E du^2 + 2F du dv + G dv^2$$

where

$$\boxed{E = r_u \cdot r_u,\ F = r_u \cdot r_v,\ G = r_v \cdot r_v}$$

Note The analogue of the first fundamental form on an abstract smooth surface $\mathcal{M}$ is called a **Riemannian metric**.

Note To find the length of a curve $\gamma = \gamma(u(t), v(t))$ on the surface, we calculate

$$\boxed{\int \sqrt{E(\frac{du}{dt})^2 + 2F\frac{du}{dt} + G(\frac{du}{dt})^2} dt}$$