Instructions: Show all of your work, and clearly indicate your answers.

Q1 Suppose f is defined on $[-1,3]$ and satisfies:

$$f(x)=\begin{cases} x, & -1\le x<0 \\ -\frac{1}{2}x^2, & 0\le x<1 \\ \sqrt{1-(x-2)^2}, & 1\le x\le 3, x\ne 2 \\ 0, & x=2 \end{cases}$$

(a) Sketch the graph of the function given above.

(b) Does $\lim_{x\to 2} f(x)$ exist? Justify your answer.

(c) Does $\lim_{x\to 1} f(x)$ exist? Justify your answer.

(d) Does $\lim_{x\to 4} f(x)$ exist? Justify your answer.

Solution:

Q2 Use Sandwich Theorem and limit laws to show that

$$\lim_{x\to 0}\left(x^2\cos(\frac{2}{x})+1\right)\left(x^2\cos(\frac{2}{x})-1\right)=-1$$

Solution:

Q3 Find the domain of the function

$$f(x)=\sqrt{1-x}+\frac{1}{\sqrt{1+x}}.$$

Solution:

Q4

1. Find the domain of the function $f(x)=\sqrt{6-2x}$.

2. Suppose that $f(x)=\frac{1}{4-x}$ and $g(x)=2^x$.

a. Find a formula for $(f\circ g)(x)$.

b. Find the domain of $(f\circ g)(x)$.

Solution:

Q5 If the point $(2,3)$ is on the graph of an odd function, then what other point must also be on the graph?

Solution:

Q6 Given that $f(x)=\frac{1}{4}\ln(2x-5)$, find a formula for $f^{-1}(x)$.

Solution:

Q7 What is the domain of the function $f(x)=\sqrt{x+9}+\sqrt{x+2}$?

1. $(-\infty,-9]$

2. $(-\infty,-2]$

3. $(-\infty,2]$

4. $(-\infty,9]$

5. $(-\infty,\infty)$

6. $[-9,\infty)$

7. $[-2,\infty)$

8. $[2,\infty)$

Solution:

Q8 If the point $(3,-6)$ is on the graph of a one-to-one function f, then which one of the following points must be on the graph of f^{-1} ?

1. $(-6, 3)$
2. $(-6, -3)$
3. $(3, 6)$
4. $(-3, 6)$
5. $(-3, -6)$
6. $(6, 3)$
7. $(6, -3)$

Solution:

Q9 If the point $(3, -6)$ is on the graph of an odd function f, then which one of the following points must also be on the graph of f ?

1. $(-6, 3)$
2. $(-6, -3)$
3. $(3, 6)$
4. $(-3, 6)$
5. $(-3, -6)$
6. $(6, 3)$
7. $(6, -3)$

Solution:

Q10 If the point $(3, -6)$ is on the graph of an even function f, then which one of the following points must also be on the graph of f ?

1. $(-6, 3)$
2. $(-6, -3)$
3. $(3, 6)$
4. $(-3, 6)$
5. $(-3, -6)$
6. $(6, 3)$
7. $(6, -3)$

Solution:

Q11 Find a formula for $f^{-1}(x)$ given that $f(x) = \ln\left(\frac{x-8}{5}\right)$.

Solution:

Q12 Suppose that $f(x)$ is an odd function, $g(x)$ is an even function, $f(5) = 3$ and $g(4) = -5$. What is the value of $(f \circ g)(-4)$.
Ans: $(f \circ g)(-4) = -3$.

Solution:

Q13 Determine the domain of the given function.

$$f(x) = \frac{x^2 - 1}{3\sqrt{2} - \sqrt{50 - 2x^2}}$$

Ans: $-5 \leq x \leq 5$ and $x \neq \pm 4$

Solution:

Q14 Circle **true** if the given statement is always true. Otherwise circle **false**.

1. If $v(t)$ is an even function and $w(t)$ is an odd function, then $p(t) = v(t)w(t)$ is an odd function.

   true or **false** ?

2. Given a function $g(x)$, if the finite limit $\lim_{x\to 9} \frac{g(x)-g(9)}{x-9}$ exists then $g(x)$ is continuous at 9.

   true or **false** ?

3. If a function $h(x)$ is not defined at $x = a$, then $\lim_{x\to a} h(x)$ does not exist.

   true or **false** ?

4. If a function $f(x)$ is one-to-one then $f(1) = 1$.

   true or **false** ?

5. A function which is continuous at a point a must also be differentiable at a.

   true or **false** ?

Solution:

Q15 Evaluate the following limits.

1. $\lim_{x\to 0} \frac{\sqrt{25+16x^2}-5}{2x^2}$

   *Ans:*4/5

2. $\lim_{x\to 8^+} \frac{\ln x}{16-2x}$

   Ans:$-\infty$

Solution:

Q16 Which one of the following equations must hold in order for a function g to be continuous at a number k ?

(a) $\lim_{x\to\infty} g(x) = g(k)$ (b) $\lim_{x\to\infty} g(x) = 0$ (c) $\lim_{x\to\infty} g(x) = k$

(d) $\lim_{x\to 0} g(x) = g(k)$ (e) $\lim_{x\to 0} g(x) = 0$ (f) $\lim_{x\to 0} g(x) = k$

(g) $\lim_{x\to k} g(x) = g(k)$ (h) $\lim_{x\to k} g(x) = 0$ (i) $\lim_{x\to k} g(x) = k$

Solution:

Q17 Find the equation of the line which goes through the point (3,-2) and is parallel to the line given by the equation $2x - 3y = 1$.

Solution:

Writing the equation of the given line as $y = \frac{2}{3}x - \frac{1}{3}$, we see it has slope $m = 2/3$. Thus, the line we seek goes through (3,-2) and has slope 2/3, so has the equation

$$\frac{y+2}{x-3} = \frac{2}{3} \quad \text{or} \quad y = \frac{2}{3}x - 4 \ .$$

Q18 Find the equation of the line which goes through the point (0,7) and is perpendicular to the line given by the equation $2x + 3y = 10$.

Solution:

The given equation can be written as $y = -\frac{2}{3}x + 10/3$. This line has slope $-2/3$, so the line we seek has slope $3/2$. Then, by the point-slope formula

$$\frac{y-7}{x-0} = \frac{3}{2} \ ,$$

which simplifies to $3x - 2y = -14$.

Solution:

Q19 Determine the domain of the given function.

$$f(x) = \frac{\sqrt{25-x^2} + \sqrt{9+\sin(x-1)}}{x^2-8x-20}$$

Ans: $[-5,-2)\bigcup(-2,5)]$

Solution:

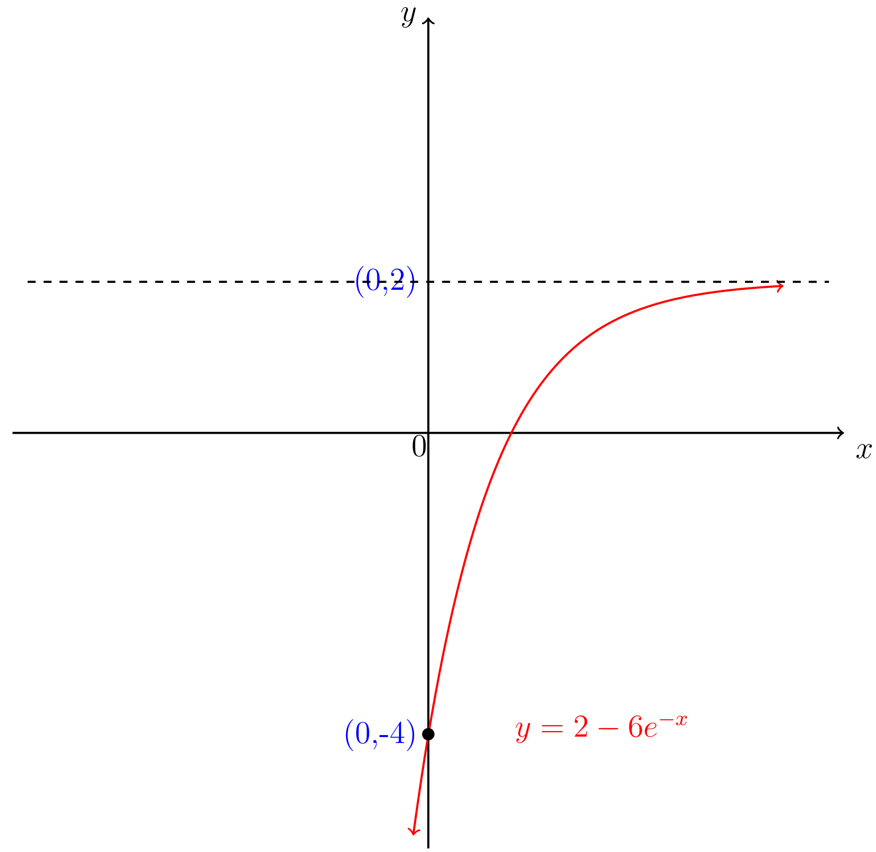
Q20 Given that $w(x) = \frac{3}{\ln(2e^x+5)}$, find a formula for $w^{-1}(x)$. *Ans*: $w^{-1}(x) = \ln\left(\frac{e^{3/x}-5}{2}\right)$

Solution:

Q21 Carefully sketch a graph of the following function. You should clearly label the value for any intercepts or asymptotes.

$$y = 2 - 6e^{-x}$$

Solution:



Q22 Suppose that $f(x) = \frac{1}{\sqrt{5-x}}$ and $g(x) = \sqrt{x-2}$. What is the domain of the composite function $(f \circ g)(x)$? *Ans:* $[2, 27]$

Solution:

Q23 For what value of the constant C is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} \frac{6x+\sin x}{2x} & \text{if} \quad x < 0 \\ e^x + C & \text{if} \quad x \geq 0 \end{cases}$$

Solution:

Ans: 5/2.

Q24 Is the following function even, odd or neither?

$$g(x) = \left(3x^2 + \cos^5 x\right)^7$$

Solution:

Q25 What is the domain of the function $f(x) = \sqrt{5 - \sqrt{2x-1}}$?

Solution:

Q26 Suppose that f is a one-to-one function which takes on the following values.

$$f(-3) = 8, f(-2) = 3, f(-1) = 1/3, f(0) = -1/3, f(1) = -3$$

,

$$f(2) = -4, f(3) = -11$$

What is the value of $f^{-1}(3)$?

Solution:

Q27 There is an odd function $f(x)$ which is continuous at all real numbers and takes on the following values.

$$f(1) = 2, \quad f(2) = -4, \quad f(3) = 5, \quad f(4) = 6, \quad f(5) = 4, \quad f(6) = -2$$

Evaluate $\lim_{x\to-4} f(x)$.

Solution:

Q28 What is the equation of the line which passes through the point $(0, 2)$ and is parallel to the line $4x + 2y = 3$?

(a) $y = -2x + 2$ (b) $y = -2x$ (c) $y = 2x + 2$ (d) $y = 0.5x + 2$ (e) $y = -0.5x + 2$

Solution:

Q29 If ball is thrown upward from a height of 100 feet with an initial velocity of 64 feet per second, its height is given by the formula

$$h = -16t^2 + 64t + 100,$$

where h is the height in feet and t is the time in seconds. What is the maximum height the ball reaches?

(a) 164 ft. (b) 144 ft. (c) 196 ft. (d) 200 ft. (e) 150 ft.

Solution:

Q30 The domain of $f(x) = \frac{x^3-1}{(x+2)(x^2+3)}$ is...

Solution:

Q31 The average rate of change of the function $y = 3x + x^2 + 5$ over the interval $[1, 1.1]$ is...

Solution:

Q32 Given

$$f(x) = \begin{cases} x+1 & x \le 1 \\ -2x+2 & x > 1 \end{cases}$$

Compute

$$\lim_{x\to 1} f(x)$$

Solution:

Ans: It does not exist

Q33 Which one of the following equations must hold in order for a function f to be continuous at a number b ?

(a) $\lim_{x\to 0} f(x) = b$ (b) $\lim_{x\to 0} f(x) = 0$ (c) $\lim_{x\to 0} f(x) = f(b)$

(d) $\lim_{x\to b} f(x) = b$ (e) $\lim_{x\to b} f(x) = 0$ (f) $\lim_{x\to b} f(x) = f(b)$

(g) $\lim_{x\to\infty} f(x) = b$ (h) $\lim_{x\to\infty} f(x) = 0$ (i) $\lim_{x\to\infty} f(x) = f(b)$

Solution:

Q34 Suppose that f and g are one-to-one functions which take on the following values.

$$f(-2)=2, \quad f(-1)=1/2, \quad f(0)=-1/2, \quad f(1)=-2, \quad f(2)=-4$$

$$g(-2)=-4, \quad g(-1)=-2, \quad g(0)=-1/2, \quad g(1)=1/2, \quad g(2)=2$$

What is the value of

$$f^{-1}\left(g^{-1}(-4)\right)$$

?

Solution:

Q35 What value of c makes the following function continuous on $(-\infty, \infty)$?

$$f(x)=\begin{cases} x^2+5 & \text{for} \quad x<2 \\ 5x+c & \text{for} \quad x \geq 2 \end{cases}$$

Solution:

x^2+5 is continuous for $x<2$ and $5x+c$ is continuous for $x>2$. At $x=2$, we must check that $\lim_{x\to 2} f(x) = f(2)$

1. $f(2) = 5.2 + c = 10 + c$

2. $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(x^2+5) = 9$

3. $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(5x+c) = 10 + c$

Solving $10 + c = 9$ gives $c = -1$.

Q36 Find a formula for $f^{-1}(x)$ given that $f(x) = \ln\left(\frac{x-8}{5}\right)$.

Solution:

Q37 If the point $(3,-6)$ is on the graph of an odd function f, then which one of the following points must also be on the graph of f ?

1. $(-6,3)$

2. $(-6,-3)$

3. $(3,6)$

4. $(-3,6)$

5. $(-3,-6)$

6. $(6,3)$

7. $(6,-3)$

Solution:

We have $f(3)=-6$. Since f is odd $f(-3)=-(-6)=6$ so $(-3,6)$ is on the graph of f.

Q38 If the point $(3,-6)$ is on the graph of an odd function f, then which one of the following points must also be on the graph of f ?

1. $(-6,3)$

2. $(-6,-3)$

3. $(3,6)$

4. $(-3,6)$

5. $(-3,-6)$

6. $(6,3)$

7. $(6,-3)$

Solution:

We have $f(3) = -6$. Since f is even $f(-3) = -6$ so $(-3, -6)$ is on the graph of f.

Q39 What is the domain of f^{-1}?

Solution: The range of f.

Q40 What is the range of f^{-1}?

Solution:

Q41 Let $f(x) = \sqrt{9-x}, \qquad g(x) = x^2$.

- Find the formula for $f \circ g$, and find the domain of $f \circ g$.

- Find the formula for $g \circ f$, and find the domain of $g \circ f$.

Solution:

- $D_{f \circ g} = [-3, 3]$.
- $D_{g \circ f} = (-\infty, 9]$.

Q42 Find

$$\lim_{h \to 0} \frac{\sqrt{4+h}-2}{h}.$$

Solution:. By inserting $h = 0$ we see that both numerator and denominator is zero. This is sometimes denoted,

$$\lim_{h \to 0} \frac{\sqrt{4+h}-2}{h} = \left[\frac{0}{0}\right].$$

, and is called a zero-over-zero expression. In general, *nothing* can be known about the limit from simply knowing that it is a zero expression. The strategy is to manipulate the expression until we can apply the limit laws. Often when you see an expression with a square root, it is a good idea to "rationalize" the fraction as follows:

$$\begin{aligned}\frac{\sqrt{4+h}-2}{h} &= \frac{(\sqrt{4+h}-2)(\sqrt{4+h}+2)}{h(\sqrt{4+h}+2)} \\ &= \frac{(4+h)-4}{h(\sqrt{4+h}+2)} \\ &= \frac{1}{\sqrt{4+h}+2} \to \frac{1}{4}\end{aligned}$$

as $h \to 0$. In the last step we used the limit law $\lim(f/g) = \lim f / \lim g$, which is permissible, since the limit of the denominator exists and is $\neq 0$.

Q43

Find the following limits:

1. $\lim_{x\to\infty} \frac{x^2}{\sqrt{9x^4+10x}}$.

2. $\lim_{x\to 0} \sin(\cos x)$.

Solution:

1.

$$\lim_{x\to\infty} \frac{x^2}{\sqrt{9x^4+10x}} = \lim_{x\to\infty} \frac{x^2}{x^2\sqrt{9+10/x^3}} \quad (1)$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{9+10/x^3}} \quad (2)$$

$$= \frac{1}{3}. \quad (3)$$

2. By continuity of sin and cos we have:

$$\lim_{x\to 0}\sin(\cos x) = \sin(\lim_{x\to 0}\cos(x)) = \sin(\cos(0)) = \sin(1).$$

Q44 Find the following limits:

1. $\lim_{x\to 0}\sin(\frac{1}{x})$ (Ans: DNE (does not exist))

2. $\lim_{x\to 0}\tan(\frac{1}{x})$ (Ans: DNE)

3. $\lim_{x\to 0}\frac{1}{\sin x}$ (Ans: DNE)

Solution: $\lim_{x\to 0^+}\frac{1}{\sin x} = \infty$ $\lim_{x\to 0^-}\frac{1}{\sin x} = -\infty$ Therefore the limit does not exist. (DNE)

4. $\lim_{x\to 0}\frac{1}{\cos x}$ (Ans: 1)

5. $\lim_{x\to \frac{\pi}{2}}\frac{x}{\cos x}$ (Ans: DNE)

6. $\lim_{x\to 0}\frac{\cos 3x}{\sin x}$ (Ans: DNE)

7. $\lim_{x\to 0}\frac{\cos x}{x}$ (Ans: DNE)

8. $\lim_{x\to 0}\frac{\sin x}{\sqrt{x}}$ (Ans: 0)

9. $\lim_{x\to 0}\frac{x^2}{1-\cos x}$ (Ans: 2)

10. $\lim_{x\to \frac{\pi}{4}}\frac{\cos 2x}{\cos x-\sin x}$ (Ans: $\sqrt{2}$)

11. $\lim_{x\to \frac{\pi}{4}}\frac{\cos 2x}{1-\sqrt{2}\sin x}$ (Ans: 2)

12. $\lim_{x\to 0}x\sin\frac{1}{x}$ (Ans: 0)

13. $\lim_{x\to \frac{\pi}{2}}\tan x$ (Ans: DNE)

14. $\lim_{x\to \frac{\pi}{2}}\frac{\cos x}{\sin 2x}$ (Ans: 1/2)

15. $\lim_{x\to 0}\frac{x}{x+\sin x}$ (Ans: 1/2)

Solution:

Q45 Consider the curve $y = f(x)$, where

$$f(x) = \frac{4x^2}{2x+1}.$$

(a) Find the domain of f.

(b) Find any x-intercepts or y-intercepts of the curve.

(c) check for symmetries of the curve $y = f(x)$.

(d) Find any vertical, horizontal, or slant asymptotes for the curve $y = f(x)$.

Solution:

(a) The function f is defined for all x except $x = -1/2$, so the domain of f is the set $(-\infty, -1/2) \cup (1/2, \infty)$.

(b) The y-intercept of the curve is where the curve crosses the y-axis (i.e., at $x = 0$), and since $f(0) = 0$,

- the y-intercept of the curve is $y = 0$.

The curve has x-intercepts for those x for which $f(x) = 0$. Since

$$f(x) = \frac{4x^2}{2x+1} = 0, \quad \text{at } x = 0,$$

it follows that

- there is one x-intercept for the curve, at $x = 0$.

(c) To check for symmetries, we recall that the curve $y = f(x)$ is symmetric about the y-axis if $f(-x) = f(x)$, and symmetric about the origin if $f(-x) = -f(x)$, for all x in the domain of f. For this particular function,

$$f(-x) = \frac{4(-x)^2}{2(-x)+1} = \frac{4x^2}{-2x+1},$$

so $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$. The curve is not symmetric about the y-axis or about the origin.

(d) The determination of vertical, horizontal, or slant asymptotes is next:

(i) Vertical asymptotes: The function f is undefined at $x = -1/2$. Then since

$$\lim_{x\to(-1/2)^-} \frac{4x^2}{2x+1} = -\infty, \quad \text{and} \quad \lim_{x\to(-1/2)^+} \frac{4x^2}{2x+1} = +\infty,$$

the curve has a vertical asymptote at $x = -1/2$.

(ii) Horizontal asymptotes: Looking at the behavior of $f(x)$ as $x \to \pm\infty$,

$$\lim_{x\to\infty} \frac{4x^2}{2x+1} = \lim_{x\to\infty} \frac{4x}{2+(1/x)} = +\infty,$$

and

$$\lim_{x\to-\infty} \frac{4x^2}{2x+1} = \lim_{x\to\infty} \frac{4x}{2+(1/x)} = -\infty.$$

so the curve does not approach a constant (i.e., a horizontal line) as $x \to \pm\infty$. Thus there are no horizontal asymptotes for this curve.

Q46 Consider the curve $y = f(x)$, where

$$f(x) = \frac{x^2+x+1}{x^2-x-2}$$

(a) Find the domain of f.

(b) Find any x-intercepts or y-intercepts of the curve.

(c) check for symmetries of the curve $y = f(x)$.

(d) Find any vertical, horizontal, or slant asymptotes for the curve $y = f(x)$.

Q47 Consider the curve $y = f(x)$, where

$$f(x) = \frac{3x-2}{x-1}$$

(a) Find the domain of f.

(b) Find any x-intercepts or y-intercepts of the curve.

(c) check for symmetries of the curve $y = f(x)$.

(d) Find any vertical, horizontal, or slant asymptotes for the curve $y = f(x)$.

Q48 Consider the curve $y = f(x)$, where

$$f(x) = \frac{x^2+2x+1}{x^2+1}$$

(a) Find the domain of f.

(b) Find any x-intercepts or y-intercepts of the curve.

(c) check for symmetries of the curve $y = f(x)$.

(d) Find any vertical, horizontal, or slant asymptotes for the curve $y = f(x)$.

Q49 Find the vertical and horizontal asymptotes of

$$f(x) = \frac{x-1}{x^2-4x-5}$$

(a) Find the domain of f.

(b) Find any x-intercepts or y-intercepts of the curve.

(c) check for symmetries of the curve $y = f(x)$.

(d) Find any vertical, horizontal, or slant asymptotes for the curve $y = f(x)$.

Q50 Find the average rate of change of the function

$$f(x) = \tan x - x$$

over the interval $[-\pi/4, \pi/4]$.

Solution: The average rate of change of a function $f(x)$ over the interval $[a, b]$ is given by the slope of the secant line joining $(a, (f(a))$ to $(b, f(b))$, or by

$$\frac{f(b)-f(a)}{b-a}.$$

So the average rate of change of $f(x) = \tan x - x$ over the interval $[-\pi/4, \pi/4]$ is given by

$$\begin{aligned} \frac{f(\pi/4)-f(-\pi/4)}{\pi/4-(-\pi/4)} &= \frac{(\tan(\pi/4)-\pi/4)-(\tan(-\pi/4)-(-\pi/4))}{\pi/2} \\ &= \frac{2}{\pi}\left[\left(1-\frac{\pi}{4}\right)-\left(-1+\frac{\pi}{4}\right)\right] \\ &= \frac{2}{\pi}\left(2-\frac{\pi}{2}\right). \end{aligned}$$

Q51 Let

$$f(x) = x^3 + 3.$$

Show that f has at least one real zero. In other words, show that $f(c) = 0$ for some real number c (you do *not* have to calculate c explicitly).

Solution: $f(-2) < 0$ and $f(0) > 0$. By IVT, f has a root in $(-2, 0)$.

Q52

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Q53

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Solution:

” We must know, we will know” (David Hilbert)