Instructions: Keep all devices capable of communication turned off and out of sight.

Question 1

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that $T\left(\begin{pmatrix}1\\-1\end{pmatrix}\right) = \begin{pmatrix}1\\2\\3\end{pmatrix}$ and $T\left(\begin{pmatrix}-2\\3\end{pmatrix}\right) = \begin{pmatrix}-2\\-4\\7\end{pmatrix}$. Find $T\left(\begin{pmatrix}x\\y\end{pmatrix}\right)$.

Ans: $T\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = \begin{pmatrix}x\\2x\\16x+13y\end{pmatrix}$.

Question 2

Describe the span in $\mathbb{R}^3$ of the vectors $\begin{pmatrix}1\\0\\3\end{pmatrix}$ and $\begin{pmatrix}-1\\1\\3\end{pmatrix}$, i.e.

$$span\left\{\begin{pmatrix}1\\0\\3\end{pmatrix}, \begin{pmatrix}-1\\1\\3\end{pmatrix}\right\} = ?$$

Ans: $span\left\{\begin{pmatrix}1\\0\\3\end{pmatrix}, \begin{pmatrix}-1\\1\\3\end{pmatrix}\right\} = -3x + z = 0$ is a plane.

Question 3

1. Rewrite the system in the matrix form $AX = B$
2. Solve the system

$$\begin{array}{rcrcrcrcr} x & - & 2y & & & + & w & = & 2 \\ & & & & -z & + & 3w & = & 0 \\ 2x & - & 4y & + & 3z & - & 7w & = & 4 \end{array}$$

3. Write the solution set in parametric vector form.

Ans: $\begin{pmatrix}1 & -2 & 0 & 1\\0 & 0 & -1 & 3\\2 & -4 & 3 & -7\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \begin{pmatrix}2\\0\\4\end{pmatrix}$.

$$\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \begin{pmatrix}2y-w+2\\y\\3w\\w\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \begin{pmatrix}2s-t+2\\s\\3t\\t\end{pmatrix} = \begin{pmatrix}2\\0\\0\\0\end{pmatrix} + s\begin{pmatrix}2\\1\\0\\0\end{pmatrix} + t\begin{pmatrix}-1\\0\\3\\1\end{pmatrix}$$

where s, t any real number. It is clear that the system has infinitely many solutions.

Question 4

Let T be a one-to-one linear transformation. Prove that if $\{u, v, w\}$ is a linearly independent set then $\{T(u), T(v), T(w)\}$ is a linearly independent set.

Proof: Suppose $\{u, v, w\}$ is a linearly independent set. If

$$c_1T(u) + c_2T(v) + c_3T(w) = 0$$

then by linearity of T we have

$$T(c_1u + c_2v + c_3w) = 0$$

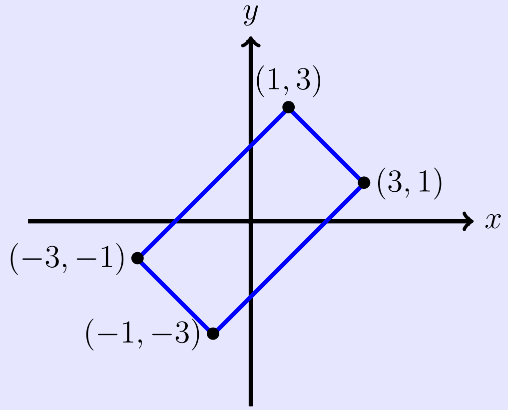
and using the fact that T is one-to-one, we have

$$c_1u + c_2v + c_3w = 0.$$

Since $\{u, v, w\}$ is a linearly independent set, we get $c_1 = c_2 = c_3 = 0$. Thus $\{T(u), T(v), T(w)\}$ is a linearly independent set.

Question 5

The following figure shows a rectangle in the plane. Find the new coordinates of the four vertices if this rectangle is rotated 45° counterclockwise around the origin.



Question 6

Is there a linear transformation that maps $(1,0)$ to $(5,3,4)$ and maps $(3,0)$ to $(1,3,2)$? Explain.

Question 7

Determine the quadratic function $p(x) = ax^2 + bx + c$ that passes through the points $(2,4), (3,6), (4,10)$..

Ans: $p(x) = x^2 - 3x + 6$

Question 8

Let c be a scalar and let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} 3 \\ 5 \\ c \end{pmatrix}$. For what values of c are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ linearly independent?

Question 9

Let v and w be vectors in $\mathbb{R}^n$. Let $S = \{v, w\}$ and $T = \{v, v - w\}$. Show that $span\, S = span\, T$.

Question 10

Write a linear system corresponding to the given augmented matrix.
$$\left(\begin{array}{ccccc|c} 6 & 2 & -3 & 4 & 1 & 0 \\ 5 & 0 & 0 & 1 & 200 & 2 \end{array}\right)$$

Question 11

Give an example of a system of three linear equations in two variables that has infinitely many solutions.

Ans:
$$\begin{array}{rcrcr} x & + & y & = & 1 \\ 3x & + & 3y & = & 3 \\ 5x & + & 5y & = & 5 \end{array}$$

Question 12

Give an example of a system of four linear equations in two variables that has a unique solution.

Ans:

Question 13

If possible, express $\begin{pmatrix} 7 \\ 7 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

Ans:

Question 14

Find the values of a, b and c such that the following system

$$\begin{array}{rcrcrcl} x & - & y & + & 2z & = & a \\ 2x & + & 3y & - & z & = & b \\ 7x & + & 3y & + & 4z & = & c \end{array}$$

(i) is inconsistent;

(ii) has infinitely many solutions;

(iii) has a unique solution.

Ans: The system is inconsistent when $c \neq 2b + 3a$. Otherwise it has infinitely many solutions.(No unique solution)

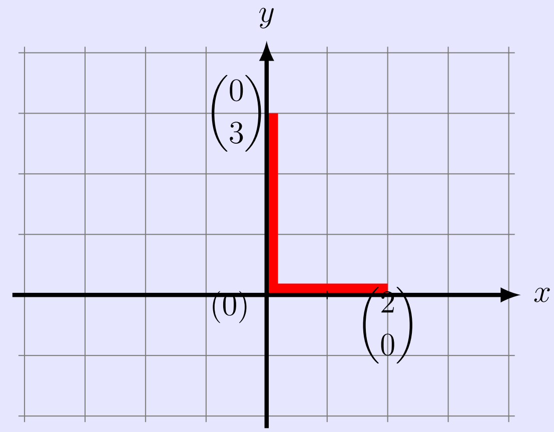
Question 15

Consider augmented matrix of a system of linear equations
$$\left(\begin{array}{ccc|c} 1 & m & 2 & 0 \\ 0 & 3 & m+2 & 0 \\ 0 & 0 & m^2-1 & (m+1)(3m-2) \end{array}\right)$$

1. For which values of m does the system have no solutions. Explain.
2. For which values of m does the system have a unique solution. Explain.
3. For which values of m does the system have a infinitely many solutions. Explain.

Question 16

Find the image of our standard letter **L** under the linear transformation $T(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.



True/False . No justifications are needed.

Q1 Subsets of linearly dependent sets are linearly dependent.

T F

Q2 Every system of linear equations has a solution.

T F

Q3 If three nonzero vectors are linearly dependent, then one of them is a scalar multiple of one of the others.

T F

Q4 The following set S is a basis for $\mathbb{R}^6$.

$$S = \{(3, 2, 0, 8, 5, 2), (4, 3, 2, 0, 4, 1), (3, 2, 1, 4, 5, 2), (2, 3, 2, 1, 0, 0)\}$$

T F

Q5 The vectors $v_1 = (3, 1, 5), v_2 = (2, 4, 3)$, and $v_3 = (5, 5, 8)$ are linearly dependent.

T F

Q6 The vectors $v_1 = (3,1,4,2), v_2 = (5,3,7,1), v_3 = (6,2,8,4)$ and $v_3 = (5,5,8)$ are linearly independent.

T F

Q7 The dimension of the set $S = \text{Span}\{\begin{pmatrix}-2\\2\end{pmatrix}, \begin{pmatrix}3\\-3\end{pmatrix}\}$ is 2.

T F

Q8 The dimension of the set $S = \text{Span}\{\begin{pmatrix}1\\-2\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\}$ is 2.

T F

Q9 The function $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (x^2, y)$ is a linear transformation.

T F

Q10 If T is a Linear transformation then $T(0) = 0$

T F

Q11 A consistent linear system with 2 equations and 3 variables must have infinitely many solutions.

T F

Q12 Homogeneous systems are always consistent.

T F

Q13 A system with more equations than variables must be inconsistent.

T F

Q14 If a consistent system has more variables than equations, it must have infinitely many solutions.

T F

Q15 A 5×6 matrix has six rows.

T F

Q16 An inconsistent system has more than one solution.

T F

Q17 Two matrices are row equivalent if they have the same number of rows.

T F

Q18 Two linear systems are equivalent if they have the same solution set.

T F

Q19 A basic variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix.

T F

Q20 The row reduction algorithm applies only to augmented matrices for a linear system.

T F

Q21 The echelon form of a matrix is unique.

T F

Q22 The reduced row echelon form of a matrix is unique.

T F

Q23 Whenever a system has free variables, the solution set contains many solutions

T F

Q24 If the equation $Ax = b$ is inconsistent, then b is not in the set spanned by the columns of A.

T F

Q25 A linear transformation preserves the operations of vector addition and scalar multiplication.

T F

Q26 Every matrix transformation is a linear transformation.

T F

Q27 If **x** and **y** are linearly independent, and if **z** is in the $Span\{x, y\}$ then $\{x, y, z\}$ is linearly dependent.

T F

Q28 Two vectors are linearly dependent if and only if they lie on a line through the origin.

T F

Q29 Every system of linear equations with more variables than equations must have infinitely many solutions.

T F

Q30 If a system of linear equations has the trivial solution then it must be a homogeneous system of equations.

T F

Q31 If a homogeneous system of linear equations is also a triangular system, then the only solution is the trivial solution.

T F

Q32 If the set of vectors $\{u_1, u_2\}$ spans $\mathbb{R}^2$, then so does the set $\{u_1, u_2, u_3\}$, for any vector u_3 in $\mathbb{R}^2$.

T F

Q33

$$\text{Span}\Bigg\{ \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix} \} = \mathbb{R}^3$$

T F