Multiple Choice Questions (60 points)

Q1 Under what condition can a vector (x, y, z) be written as a linear combination of $(1, -1, 1)$ and $(2, 1, 0)$.

(A) $x + y + z = 0$ | (C) $2x - y - z = 0$ | (E) None

(B) $3x + 2x + z = 0$ | (D) $x - 2y - 3z = 0$

Q2 If $A = \begin{pmatrix} 3 & 1 & -5 \\ 0 & 1 & -5 \\ 0 & -2 & 10 \end{pmatrix}$. What is $rank(A)$?

(A) 3 | (B) 2 | (C) 1 | (D) 4 | (E) None

Q3 For the augmented matrix $A = \left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right)$, find the number of free parameters.

(A) 0 | (B) 1 | (C) 2 | (D) 3 | (E) None

Q4 If the coefficient matrix A in a **homogeneous system** of 12 equations in 16 unknowns is known to have $rank(A) = 5$, how many free parameters are there in the general solution?

(A) 2 | (B) 3 | (C) 7 | (D) 11 | (E) None

Q5 What is the product : $\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

(A) $(3, 1)$ | (C) $(2, 6)$ | (E) None

(B) $(1, 7)$ | (D) $(7, 9)$

Q6 The Span$\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$ is

(A) $y = x + 1$ | (C) $y = 4x$ | (E) None

(B) $y = -2x$ | (D) $y = 1$

Q7 Find the value of p such that the linear system $\begin{cases} x - y = 3 \\ x + py = p \end{cases}$ has no solution.

(A) $p = -1$ | (C) $p = 1$ | (E) None

(B) $p = 0$ | (D) $p = 2$

Q8 If the augmented matrix $[A|b]$ of a system $Ax = b$ is row equivalent to $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ then the solution to system is

(A) $(5, -3, 1)$ | (C) $(0, 0, 0)$ | (E) None

(B) $(3, -4, 5)$ | (D) $(5, -2, 1)$

Q9 Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Which of the following is the correct geometric interpretation of the associated linear transformation?

(A) shear | (C) projection | (E) None

(B) reflection | (D) rotation

Q10 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear mappting, satisfying $T(1, 2) = (2, 0, 1)$ and $T(2, 3) = (0, 1, 2)$. Find $T(1, 0)$.

(A) $(1, 2, 3)$ | (C) $(2, 0, 1)$ | (E) None

(B) $(-1, -6, 2)$ | (D) $(-6, 2, 1)$

Classical Problems: Show all work. No work=No credit(40 pts)

Q2(20pts) The following is the reduced row-echelon form of the augmented matrix of a system of linear equations. $\left(\begin{array}{ccccc|c} 1 & 0 & -3 & 0 & 6 & 8 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 5 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

1. How many equations does the system have?

   Ans: 5

2. How many variables does the system have?

   Ans: 5

3. Is this system homogeneous or non-homogeneous?

   Ans:non-homogeneous

4. Find the pivot columns.

Ans: col1, col2, col3

5. Find the basic variables.

Ans: x_1, x_2, x_4

6. Find the free variables.

Ans: x_3 and x_5 are free variables

7. What is the rank of this matrix?

Ans: 3

8. How many parameters (if any) do we need for the general solution?

Ans: 2

9. What is the system of equations corresponding to this matrix?

Ans: The corresponding system of linear equations is

$$\begin{aligned} x_1 - 3x_3 + 6x_5 &= 8 \\ x_2 + \ \ x_3 + 2x_5 &= 0 \\ x_4 + 5x_5 &= -5 \end{aligned}$$

10. What is the general solution?

Ans:

$$\begin{aligned} x_1 &= 3x_3 - 6x_5 + 8 \\ x_2 &= -x_3 - 2x_5 \\ x_3 &= x_3 \\ x_4 &= -5x_5 - 5 \\ x_5 &= t \end{aligned}$$

Let $x_3 = s$ and $x_5 = t$ then

$$\begin{aligned} x_1 &= 3s - 6t + 8 \\ x_2 &= -s - 2t \\ x_3 &= s \\ x_4 &= -5t - 5 \\ x_5 &= t \end{aligned}$$

Q1(20pts) Find all values of k so that the system:
$$\begin{aligned} x - 2y + \quad\ \ 3z &= 1 \\ 2x + ky + \quad\ \ 6z &= 6 \\ -x + 3y + (k-3)z &= 0 \end{aligned}$$
has

1. no solutions

2. a unique solution

3. infinitely many solutions

Solution: Solution: First, we convert this linear system to an augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 2 & k & 6 & 6 \\ -1 & 3 & k-3 & 0 \end{array}\right)$$

and apply row operations: $R_2 \to -2R_1 + R_2$ $R_3 \to R_1 + R_3$ $R_3 \to R_2$ $R_2 \to R_3$ $R_3 = -(4+k)R_2 + R_3$
we get

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & k & 1 \\ 0 & 0 & -k(k+4) & -k \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Case1: $k = 0$. Then the system is

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and it's consistent. There is a col without a pivot (the third) so there are infinitely many solutions.
Case2: $k = -4$. Then the system is

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and is inconsistent. Case3: $k \neq 0, 4$. Then the system is

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & k & 1 \\ 0 & 0 & nonzero & 0-k \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and there is unique solution.